One approach to the reservoir simulation in the presence of wells with given total flow rates

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Outline

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1. Original problem and preliminaries

\[ \Pi \subset \mathbb{R}^d \] bounded connected domain \( (d = 2, 3) \)

\[ \Gamma = \partial \Pi \] piece-wise smooth boundary

\[ \Omega_l, \quad l = 1, \ldots, L \] subdomains, \( \Gamma_l = \partial \Omega_l, \quad \overline{\Omega}_l \subset \Pi \)

\[ \Omega = \Pi \setminus \bigcup_{l=1}^L \overline{\Omega}_l, \quad \overline{\Omega}_l \cap \overline{\Omega}_k = \emptyset, \quad l \neq k, \quad \partial \Omega = \Gamma \cup \bigcup_{l=1}^L \Gamma_l \]

\[ \nabla \cdot a(x) \nabla p = 0, \quad x \in \Omega \]

\[ p \big|_{\Gamma} = 0 \]

\[ p \big|_{\Gamma_l} = c_l, \quad \int_{\Gamma_l} a \nabla p \cdot n \, d\gamma = Q_l, \quad l = 1, \ldots, L \]
1. Original problem and preliminaries

“Classical approach”: Auxiliary problems

\[ \nabla \cdot a(x) \nabla p^{(k)} = 0, \quad p^{(k)} |_{\Gamma} = 0, \quad p^{(k)} |_{\Gamma_l} = \delta_{kl}, \quad k, l = 1, \ldots, L \]

\[ Q_l^{(k)} = \int_{\Gamma_l} a \nabla p^{(k)} \cdot n \, d\gamma \quad \iff \quad \sum_{k=1}^{L} Q_l^{(k)} \alpha_k = Q_l \quad \implies \quad p = \sum_{k=1}^{L} \alpha_k p^{(k)} \]

It is necessary to solve \( L \) auxiliary problems

**Main idea:** use of fictitious domain technique.

It is difficult to extend functions from \( H^1(\Omega) \) into wells by constant, but it is trivially for functions from \( L_2(\Omega) \)

We need weak formulation in which \( p \in L_2(\Omega) \)
2. Fictitious domain method and mixed FEM

\[ L_{2,c}(\Pi) = \{ q \in L_2(\Pi), \quad q|_{\Omega_l} = \text{any constants}, \quad l = 1, \ldots, L \} \]

\[ \mathbf{H}_{\text{div},c}(\Pi) = \{ \mathbf{v} | \mathbf{v} \in (L_2(\Pi))^d, \quad \nabla \cdot \mathbf{v} \in L_{2,c}(\Pi) \} \]

Find \( p \in L_{2,c}(\Omega) \) and \( \mathbf{u} \in \mathbf{H}_{\text{div},c}(\Omega) \) that

\[ \forall \mathbf{v} \in \mathbf{H}_{\text{div},c}(\Pi) \quad \int_{\Omega} \frac{1}{a} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{x} - \int_{\Pi} p \nabla \cdot \mathbf{v} \, d\mathbf{x} = 0 \]

\[ \forall q \in L_{2,c}(\Pi) \quad - \int_{\Pi} q \nabla \cdot \mathbf{u} \, d\mathbf{x} = \sum_{l=1}^{L} Q_l q_l \]

**Statement 1.** Solution of \( \ast \) exists, but not unique (up to solenoidal in \( \Omega_l \) functions).

**Statement 2.** \( \ast \) and \( \ast \) are “equivalent” (in some sense).
2. Fictitious domain method and mixed FEM

Computational mesh

\[ \Pi = (0,1)^2 \quad K = [x_i, x_{i+1}] \times [y_j, y_{j+1}] \quad i, j = 0, \ldots, N - 1 \]

\[ \Omega_l \subset K_l \setminus \partial K_l \]

\[ \text{mes} (\Omega_l) = O(h^2) \]

\[ \text{mes} (K_l \setminus \Omega_l) = O(h^2) \]

Approximation by $\mathbf{RT}_{[0]}$ elements

\[ \mathbf{V}_h = V_{h,x} \times V_{h,y} \quad V_{h,x} = \text{span}\{ \varphi_{x,i} \}_{i=0}^{N} \quad V_{h,y} = \text{span}\{ \varphi_{y,j} \}_{j=0}^{N} \]

\[ \mathbf{W}_h = \{ q^h \mid q^h(x) = \sum_{l=1}^{M} q_l \chi_{K_l}(x) \} \]
2. Fictitious domain method and mixed FEM

Important properties

\[ W_h \subset L_{2,c}(\Pi) \quad \text{and} \quad V_h \subset H_{\text{div},c}(\Pi) \quad \nabla \cdot V_h = W_h \]

Mixed FEM

Find \( p^h \in W_h \) and \( u^h \in V_h \) that

\[ \forall v^h \in V_h \quad \int_{\Omega} \frac{1}{a} u^h \cdot v^h \, dx - \int_{\Pi} p^h \nabla \cdot v^h \, dx = 0 \]

\[ \forall q^h \in W_h \quad -\int_{\Pi} q^h \nabla \cdot u^h \, dx = \sum_{l=1}^{L} Q_l q_l \]

Statement 3. There exists unique solution of problem
3. Approximation results for single-phase liquid

Error estimations

\[
\| p - p^h \|_{L^2(\Omega)} \leq C \left( h \sum_{l=1}^{L} Q_l^2 \right)^{1/2}
\]

\[
\| u - u^h \|_{L^2(\Omega')} \leq C \left( h \sum_{l=1}^{L} Q_l^2 \right)^{1/2}
\]

\(\Omega' \subset \Omega\), \(\text{dist} \{ \partial \Omega', \Gamma_1 \cup \cdots \cup \Gamma_L \}\) does not depend on \(h\),

\(C\) does not depend on \(h \) and \(\Omega_i\).
3. Approximation results for single-phase liquid

Numerical examples

Comparison vs “classical approach”

$L = 2$: one injection well and one producing well

production rates: $Q_w = 1$, $Q_o = -1$

$$p = \frac{(Q_w^{(o)} + Q_o^{(o)}) p^{(w)} - (Q_w^{(w)} + Q_o^{(w)}) p^{(o)}}{Q_w^{(w)} Q_o^{(o)} - Q_w^{(o)} Q_o^{(w)}}$$

$p^{(k)}$ and $Q_l^{(k)}$, $k, l = w, o$ are calculated with usage free libraries 

Gmsh  and FEniCS
3. Approximation results for single-phase liquid

Numerical results

\begin{align*}
\begin{array}{|c|c|c|c|}
\hline
r_w & h & 3.70 \cdot 10^{-2} & 1.23 \cdot 10^{-2} & 4.12 \cdot 10^{-3} \\
\hline
6.17 \cdot 10^{-3} & 0.0159 & N/A & N/A \\
2.06 \cdot 10^{-3} & 0.0238 & 0.0050 & N/A \\
6.90 \cdot 10^{-4} & 0.0291 & 0.0067 & 0.0017 \\
\hline
\end{array}
\end{align*}

$L_2$-norm of difference when compared to the “classical approach”

N/A (Not Available): violation of inequality $2r_w/h < 1$.

Numerical error $O(h^{3/2})$

A priori error $O(h^{1/2})$
3. Approximation results for single-phase liquid

Different well locations

Pressure field

Isolines

Velocity directions field
4. Anisotropic mixed FEM

Notations:

\[ \Pi = \Pi^{(xy)} \times (0, l_z), \quad \Pi^{(xy)} \subset \mathbb{R}^2 \]

\[ \Omega = \Omega^{(xy)} \times (0, l_z), \quad \Omega^{(xy)} \subset \mathbb{R}^2 \]

wells: \[ \Omega_l = \Omega_l^{(xy)} \times (0, l_z), \quad \Gamma_l^{(xy)} = \partial \Omega_l^{(xy)} \]

\[ \nabla^{(xy)} \equiv \left( \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \right)^t \]
4. Anisotropic mixed FEM

Original problem:

\[ \nabla \cdot a(x) \nabla p = 0, \quad x \in \Omega \]

\[ p \big|_{\Gamma} = 0 \]

\[ p \big|_{\Gamma_l^{(xy)}} = c_l(z), \quad \int_{\Gamma_l^{(xy)}} a \nabla^{(xy)} p \cdot n^{(xy)} \, d\gamma = Q_l(z), \quad z \in (0, l_z), \quad l = 1, \ldots, L \]

Anisotropic mixed formulation:

\[ u^{(xy)} = -a \nabla^{(xy)} p, \quad \nabla^{(xy)} \cdot u^{(xy)} - \frac{\partial}{\partial z} \left( a \frac{\partial p}{\partial z} \right) = 0, \quad x \in \Omega \]

\[ \int_{\Gamma_l^{(xy)}} u^{(xy)} \cdot n^{(xy)} \, d\gamma = -Q_l(z), \quad z \in (0, l_z) \]
4. Anisotropic mixed FEM

\[ L_{2,c}(\Pi) = \{ q \in L_2(\Pi), \quad q \big|_{\Omega_l^{(xy)}} = \text{any constants}, \quad l = 1, \ldots, L \} \]

\[ H^{(xy)}_{\text{div}, c}(\Pi) = \left\{ \mathbf{v}^{(xy)} \left| \begin{array}{l}
\mathbf{v}^{(xy)} \in (L_2(\Pi))^2, \\
\nabla^{(xy)} \cdot \mathbf{v}^{(xy)} \in L_{2,c}(\Pi)
\end{array} \right. \right\} \]

\[ H_c^{(0,0,1)}(\Pi) = \left\{ q \big| q \in L_{2,c}(\Pi), \quad q_z \in L_2(\Pi) \right\} \]

\[ H_{c,0}^{(0,0,1)}(\Pi) = \left\{ q \big| q \in H_c^{(0,0,1)}(\Pi), \quad q(\cdot, \cdot, 0) = q(\cdot, \cdot, l_z) = 0 \right\} \]

Weak anisotropic mixed formulation

\[ \forall \mathbf{v}^{(xy)} \in H^{(xy)}_{\text{div}, c}(\Pi) \quad \int_{\Omega} \frac{1}{a} \mathbf{u}^{(xy)} \cdot \mathbf{v}^{(xy)} \, dx - \int_{\Pi} p \nabla^{(xy)} \cdot \mathbf{v}^{(xy)} \, dx = 0 \]

\[ \forall q \in H_{c,0}^{(0,0,1)}(\Pi) \quad -\int_{\Pi} q \nabla^{(xy)} \cdot \mathbf{u}^{(xy)} \, dx - \int_{\Omega} a \frac{\partial p}{\partial z} \frac{\partial q}{\partial z} \, dx = \sum_{l=1}^{L} \int_{0}^{l_z} Q_l q_l \, dz \]
4. Anisotropic mixed FEM

Mixed FEM

Anisotropic mixed FEM

Numerical example

Isolines, $y=1/2$
4. Anisotropic mixed FEM

Different cross-sections

Isolines

$Z = 1/32$

$Z = 1/16$

$Z = 3/32$

$Z = 1/8$

Velocity directions field
5. Wells problem for double-phase liquid

**Oil-water reservoir**

(Buckley-Leverett model)

\[ u = -k(S) \nabla p \]  Darcy law for the total velocity

\[ \nabla \cdot u = 0 \]  double-phase incompressible liquid mass conservation law

\[ u_w = \frac{k_w(S)}{k(S)} u \]  water velocity

\[ \phi \frac{\partial S}{\partial t} + \nabla \cdot u_w = 0 \]  water mass conservation law

**Boundary conditions**

\( l = 1, \ldots, L_w \) numbers of injection wells

\( l = L_w + 1, \ldots, L \) numbers of producing wells

\[ p \mid_{\Gamma} = 0, \quad p \mid_{\Gamma_l} = c_l, \quad \int_{\Gamma_l} (u \cdot n) \, d\gamma = -Q_l, \quad l = 1, \ldots, L \]

\[ (u \cdot n) \mid_{\Gamma_l} = (u_w \cdot n) \mid_{\Gamma_l}, \quad l = 1, \ldots, L_w \]
5. Wells problem for double-phase liquid

Mixed formulation

Find \( u \in H_{div,c}(\Pi) \), \( p, S \in L_{2,c}(\Pi) \), \( r \in H^{-1}(\Pi) \) such that

\[
\forall v \in H_{div,c}(\Pi) \quad \int_{\Omega} \frac{1}{k(S)} u \cdot v \, dx - \int_{\Pi} p \nabla \cdot v \, dx = 0
\]

\[
\forall q \in L_{2,c}(\Pi) \quad -\int_{\Pi} q \nabla \cdot u \, dx = \sum_{l=1}^{L} Q_l q_l
\]

\[
\forall q \in H^1(\Pi) \quad <r, q>_{L_2(\Pi)} + \int_{\Pi} \frac{k_w(S)}{k(S)} u \cdot \nabla q \, dx = 0
\]

\[
\forall q \in L_{2,c}(\Pi) \quad \int_{\Omega} \phi \frac{\partial S}{\partial t} q \, dx + <r, Eq>_{L_2(\Pi)} = -\sum_{l=1}^{L_w} Q_l q_l
\]
5. Wells problem for double-phase liquid

**IMPES like method**

\[
\forall \mathbf{v} \in \mathbf{V}_h \quad \int_{\Omega} \frac{1}{k(S^{n-1})} \mathbf{u}^n \cdot \mathbf{v} \, d\mathbf{x} - \int_{\Pi} p^n \nabla \cdot \mathbf{v} \, d\mathbf{x} = 0
\]

\[
\forall q \in \mathbf{W}_h \quad -\int_{\Pi} q \nabla \cdot \mathbf{u}^n \, d\mathbf{x} = \sum_{l=1}^{L} Q_l q_l
\]

\[
\forall q \in \mathbf{V}_h \quad \int_{\Pi} r^n P_h q \, d\mathbf{x} + \int_{\Pi} \frac{k_w(S^{n-1})}{k(S^{n-1})} \mathbf{u}^n \cdot \nabla q \, d\mathbf{x} = 0 \quad \text{P-G step}
\]

\[
\forall q \in \mathbf{W}_h \quad \int_{\Omega} \phi \frac{S^n - S^{n-1}}{\Delta t} q \, d\mathbf{x} + \int_{\Pi} r^n q \, d\mathbf{x} = -\sum_{l=1}^{L_w} Q_l q_l
\]
5. Wells problem for double-phase liquid

Space $V_h$ design

2D example:

$V_h$ space of piecewise bilinear functions

$P_h : V_h \rightarrow W_h$ lumping operator

![Diagram showing trial function support and lumped trial function support]
5. Wells problem for double-phase liquid

**Numerical example:** \( \Delta t = 0.01, \ h = 1/16 \) (coarse mesh)

\( S(0, x, y) = \exp\left(-10\sqrt{(x - x_0)^2 + (y - y_0)^2}\right), \quad \text{well location: } x_0 = y_0 \approx 0.47 \)

- Pressure isolines at \( t=2 \), \( t=4 \), and \( t=6 \)
- Water saturation isolines at \( t=2 \), \( t=4 \), and \( t=6 \)
6. Conclusions and outlook

- An approach to the solution of the problem with integral well rates is proposed, which does not require the solution of any auxiliary problems.

- Anisotropic mixed FEM is formulated to describe wells in the 3D case. The method can be considered for a wide class of elliptic problems.

- In contrast to the case of a single-phase liquid, at the present time there is no theoretical analysis in the nonstationary case. In addition, a more detailed experimental verification of the approach for multiphase liquids is required.

- It is important to indicate class of problems, where the application of anisotropic mixed FEM is efficient.

Main references


Thank you for attention!

Q & A