New conformal map for the Sinc approximation for exponentially-decaying functions over the semi-infinite interval

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July 5, 2018

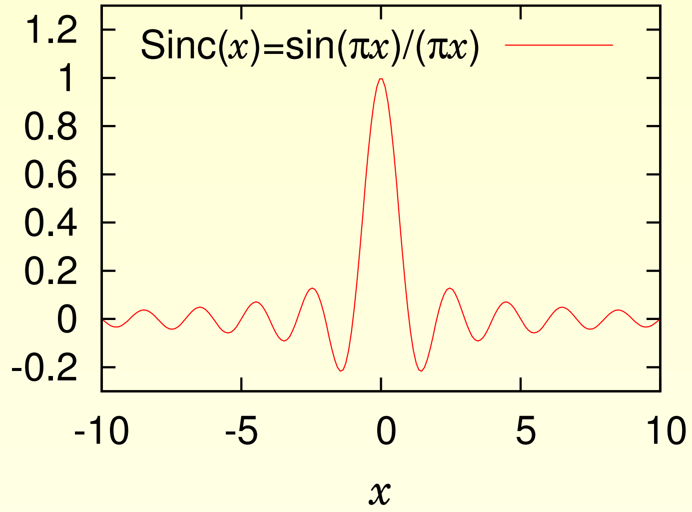
Numerical Analysis and Scientific Computation with Applications 2018

1 Sinc approximation combined with conformal map (Introduction & summary)

Sinc approximation on $\mathbb{R}$

$$F(x) \approx \sum_{j=-N}^{N} F(jh)\mathrm{Sinc}(x/h - j)$$

$$\mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Exponential convergence $\mathrm{O}(\exp(-c\sqrt{N}))$ can be attained if:

- F is defined on the whole real line ($x \in \mathbb{R}$)
- $|F(x)|$ decays quickly (exponentially) as $x \to \pm\infty$

Question What should a user do if F does not satisfy them?

$\rightarrow$ Employ a suitable variable transformation

Four typical cases that Stenger [1] considered

Let g be a smooth and bounded function and $\mu > 0$

1. $I_1 = (-\infty, \infty)$ $\quad f(t) = \left(\dfrac{1}{1+t^2}\right)^{\mu} g(t)$ polynomial decay

2. $I_2 = (0, \infty)$ $\quad f(t) = \left(\dfrac{t}{1+t^2}\right)^{\mu} g(t)$ polynomial decay

3. $I_3 = (0, \infty)$ $\quad f(t) = \left(\dfrac{t}{1+t}\right)^{\mu} \mathrm{e}^{-\mu t} g(t)$ exponential decay

4. $I_4 = (a, b)$ $\quad f(t) = \{(t-a)(b-t)\}^{\mu} g(t)$ polynomial decay

Conformal maps ψ for the 4 cases

Variable transformation $t = \psi(x)$ (Stenger [1])

$$I_1 = (-\infty, \infty) \qquad f(t) = \left(\frac{1}{1+t^2}\right)^{\mu} g(t)$$

$$t = \psi(x) = \sinh x$$

$$I_2 = (0, \infty) \qquad f(t) = \left(\frac{t}{1+t^2}\right)^{\mu} g(t)$$

$$t = \psi(x) = \mathrm{e}^{x}$$

$$I_3 = (0, \infty) \qquad f(t) = \left(\frac{t}{1+t}\right)^{\mu} \mathrm{e}^{-\mu t} g(t)$$

$$t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^{x})$$

$$I_4 = (a, b) \qquad f(t) = \{(t-a)(b-t)\}^{\mu} g(t)$$

$$t = \psi(x) = \tfrac{b-a}{2} \tanh(\tfrac{x}{2}) + \tfrac{b+a}{2}$$

Conformal maps ψ for the 4 cases

Variable transformation $t = \psi(x)$ (Stenger [1])

$I_1 = (-\infty, \infty)$

$t = \psi(x) = \sinh x$

$$f(t) = \left(\frac{1}{1+t^2}\right)^{\mu} g(t)$$

$I_2 = (0, \infty)$

$t = \psi(x) = \mathrm{e}^{x}$

$$f(\psi(x)) = \left(\frac{\mathrm{e}^{x}}{1+\mathrm{e}^{2x}}\right)^{\mu} g(\mathrm{e}^{x})$$

$\rightarrow$ put $F(x) = f(\psi(x))$

$I_3 = (0, \infty)$

$t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^{x})$

$$f(t) = \left(\frac{t}{1+t}\right)^{\mu} \mathrm{e}^{-\mu t} g(t)$$

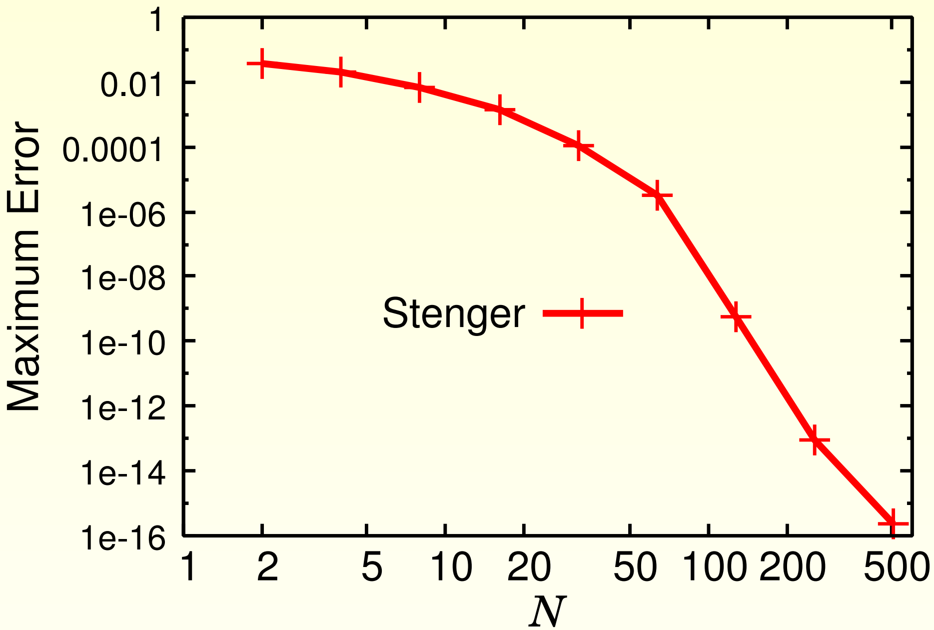
$I_4 = (a, b)$

$t = \psi(x) = \frac{b-a}{2}\tanh(\frac{x}{2}) + \frac{b+a}{2}$

$$f(t) = \{(t-a)(b-t)\}^{\mu} g(t)$$

Numerical example (implemented in C)

Case 3 $$f(t) = \sqrt{\frac{t}{1+t^2}}\,\mathrm{e}^{-t/2}, \quad t \in (0, \infty)$$

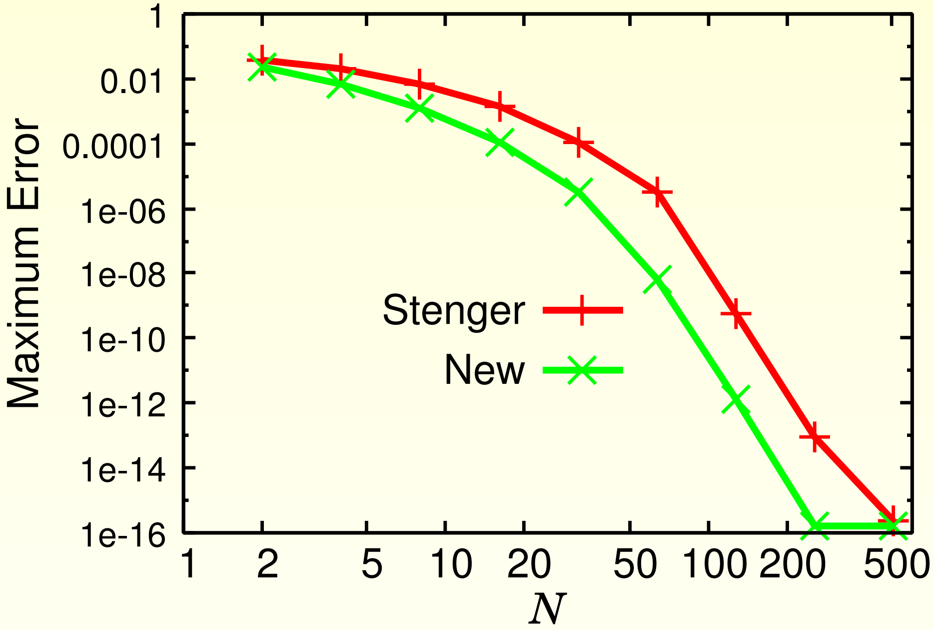


Stenger $t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^{x})$

- Maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- Exponential convergence $\mathrm{O}(\exp(-\frac{\pi}{2}\sqrt{N}))$ is obtained

Contribution 1: improve the conformal map ψ

Case 3 $\quad f(t) = \sqrt{\dfrac{t}{1+t^2}}\,\mathrm{e}^{-t/2}, \quad t \in (0,\infty)$



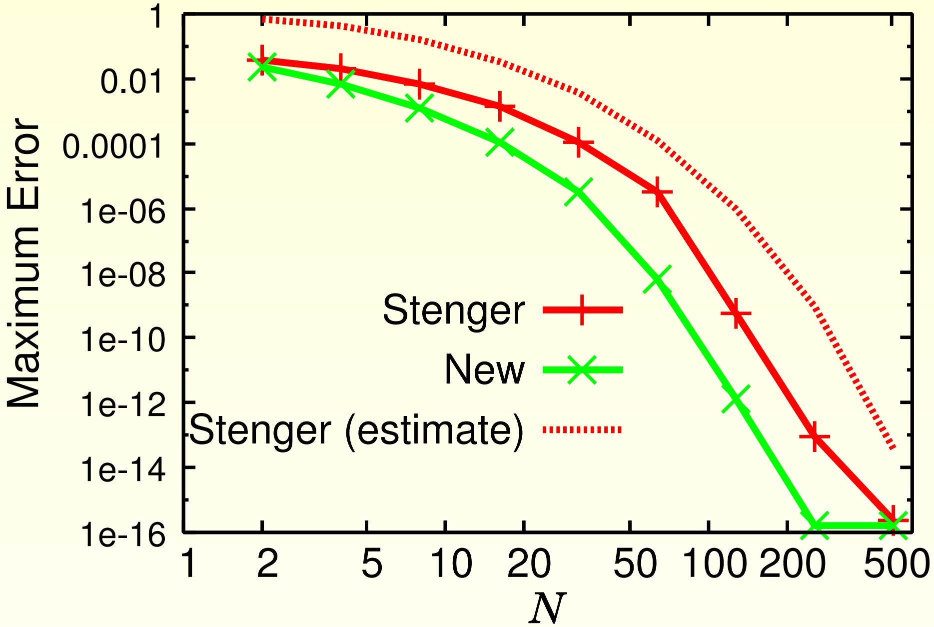
Stenger $t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^x)$

New $t = \phi(x) = \log(1 + \mathrm{e}^x)$

- Maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- Faster convergence: $\mathrm{O}(\exp(-\frac{\pi}{2}\sqrt{N})) \to \mathrm{O}(\exp(-\frac{\pi}{\sqrt{2}}\sqrt{N}))$

Error bound has been given in the case of ψ

Case 3 $f(t) = \sqrt{\dfrac{t}{1+t^2}}\,\mathrm{e}^{-t/2}, \quad t \in (0,\infty)$



Stenger $t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^x)$

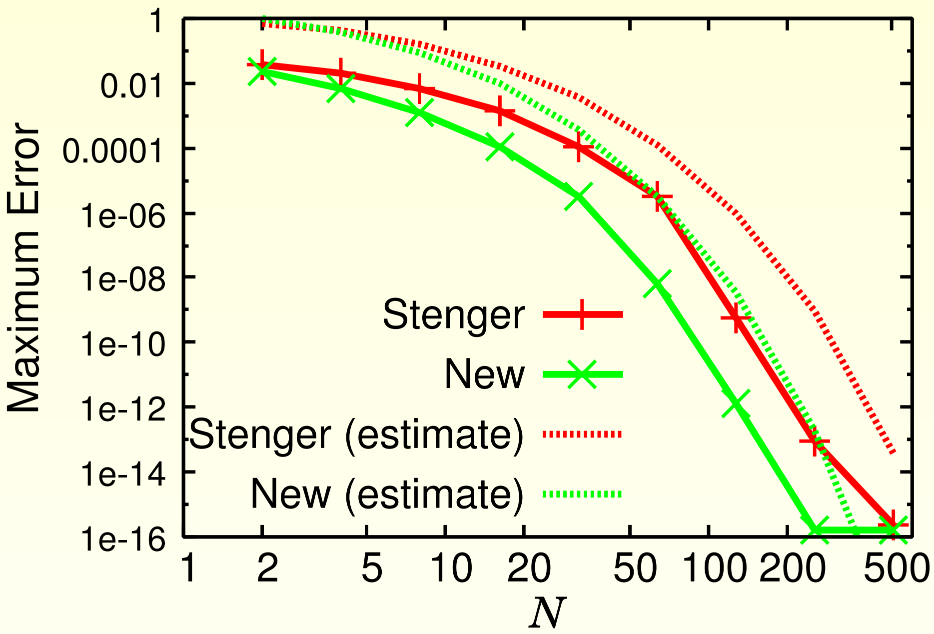
$|\text{Error}| \leq 4.662\sqrt{N}\exp(-\frac{\pi}{2}\sqrt{N})$

New $t = \phi(x) = \log(1+\mathrm{e}^x)$

- Maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- Faster convergence: $\mathrm{O}(\exp(-\frac{\pi}{2}\sqrt{N})) \to \mathrm{O}(\exp(-\frac{\pi}{\sqrt{2}}\sqrt{N}))$

Contribution 2: Error bound is given for ϕ

Case 3 $$f(t) = \sqrt{\frac{t}{1+t^2}}\,\mathrm{e}^{-t/2}, \quad t \in (0,\infty)$$



Stenger $t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^x)$

$$|\text{Error}| \leq 4.662\sqrt{N}\exp(-\tfrac{\pi}{2}\sqrt{N})$$

New $t = \phi(x) = \log(1+\mathrm{e}^x)$

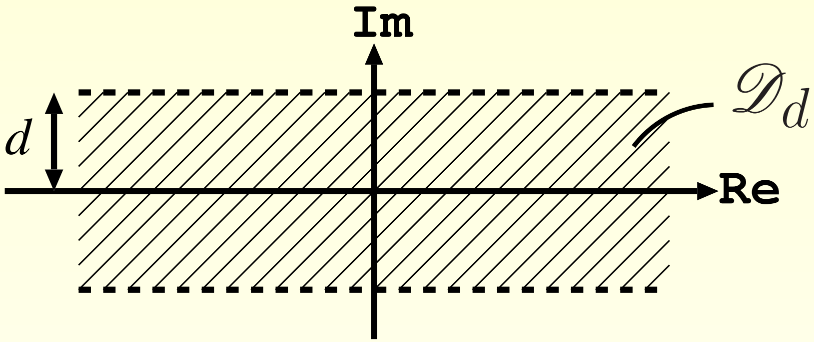
$$|\text{Error}| \leq 14.56\sqrt{N}\exp(-\tfrac{\pi}{\sqrt{2}}\sqrt{N})$$

- Maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- Faster convergence: $\mathrm{O}(\exp(-\frac{\pi}{2}\sqrt{N})) \to \mathrm{O}(\exp(-\frac{\pi}{\sqrt{2}}\sqrt{N}))$

2 Improvement of the conformal map and giving the error bound

Definition: the strip domain $\mathscr{D}_d$

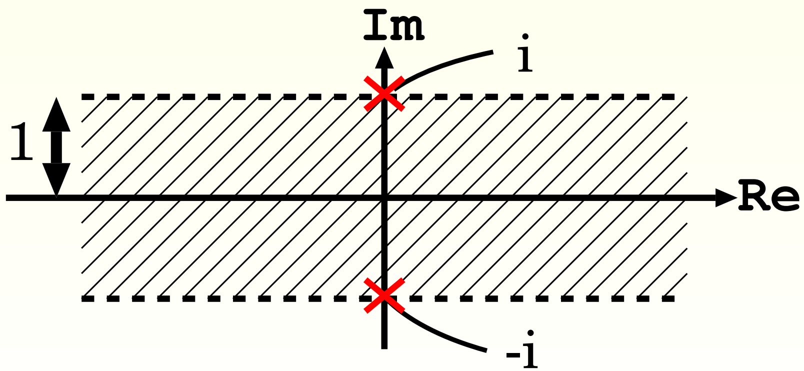
The transformed function $F(x) = f(\psi(x))$ should be analytic on the complex domain $\mathscr{D}_d$:



$$\mathscr{D}_d = \{z \in \mathbb{C} : |\operatorname{Im} z| < d\}$$

($\Rightarrow$ |Error| $= \mathrm{O}(\exp(-\pi d/h))$)

Example $F(x) = \dfrac{1}{1+x^2}$ is analytic on $\mathscr{D}_d$ with $d = 1$



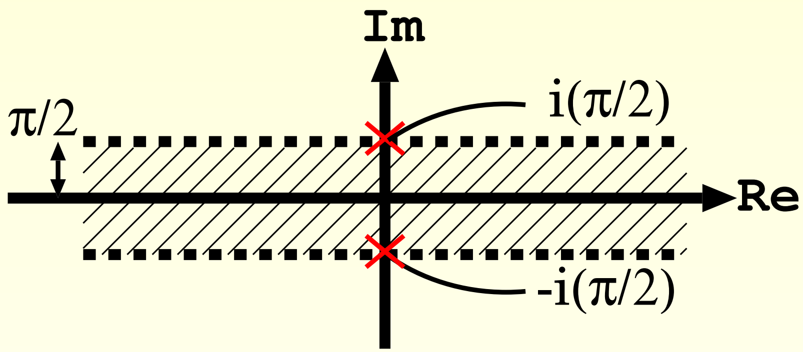
Idea of the improvement

$F'(x) = f'(\psi(x))\psi'(x)$

Existing conformal map ψ

$$\psi'(x) = \frac{1}{\sqrt{1+\mathrm{e}^{-2x}}}$$

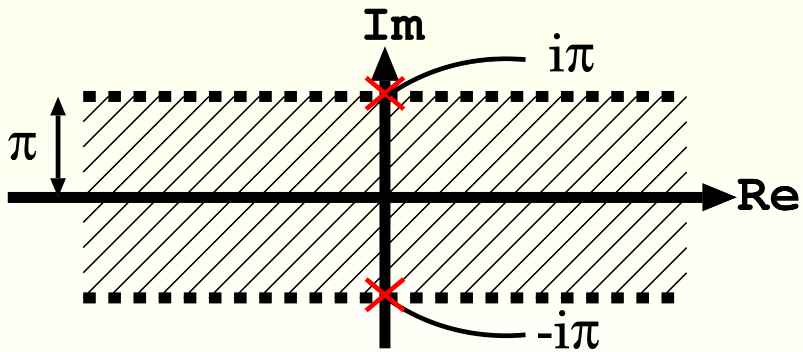
(analytic on $\mathscr{D}_d$ with $d = \pi/2$)



New conformal map ϕ

$$\phi'(x) = \frac{1}{1+\mathrm{e}^{-x}}$$

(analytic on $\mathscr{D}_d$ with $d = \pi$)



⇒ Higher convergence rate is expected (cf. $\mathrm{O}(\exp(-\pi d/h))$)

Error bound in the case of ψ (Existing result)

Existing conformal map: $\psi(x) = \operatorname{arcsinh}(\mathrm{e}^x)$

Theorem (Okayama, 2018)

- $f(\psi(\cdot))$ is analytic on $\mathscr{D}_d$. $(0 < d \leq \pi/2)$
- $\forall z \in \psi(\mathscr{D}_d)$, $|f(z)| \leq K \left|\frac{z}{1+z}\right|^{\mu} \left|\mathrm{e}^{-z}\right|^{\mu}$.

$$\Longrightarrow \quad \sup_{t\in(0,\infty)} |\mathrm{Error}(t)| \leq C\sqrt{N}\exp(-\sqrt{\pi d \mu N}),$$

where

$$C = \frac{2K}{\sqrt{\pi d \mu}} \left\{ \frac{2 \cdot 2^{\mu}}{\sqrt{\pi d \mu}(1 - \mathrm{e}^{-2\sqrt{\pi d \mu}}) \cos^{2\mu}(d/2)} + 1 \right\}.$$

Error bound in the case of ϕ (New!)

New conformal map: $\phi(x) = \log(1 + \mathrm{e}^x)$

Theorem (This work)

- $f(\phi(\cdot))$ is analytic on $\mathscr{D}_d$. $(0 < d < \pi)$
- $\forall z \in \phi(\mathscr{D}_d)$, $|f(z)| \leq K \left|\frac{z}{1+z}\right|^\mu |\,\mathrm{e}^{-z}\,|^\mu$.

$$\Longrightarrow \sup_{t\in(0,\infty)} |\mathrm{Error}(t)| \leq C\sqrt{N}\exp(-\sqrt{\pi d \mu N}),$$

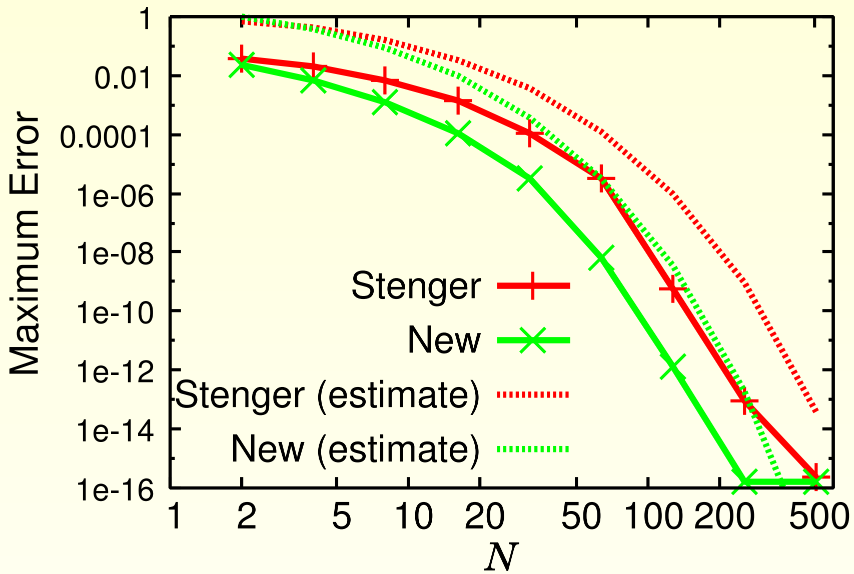
where

$$C = \frac{2K}{\sqrt{\pi d \mu}}\left\{\frac{2\cdot\{\mathrm{e}/(\mathrm{e}-1)\}^{\mu/2}}{\sqrt{\pi d\mu}(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\cos^{2\mu}(d/2)} + 1\right\}.$$

3 Numerical examples

Numerical example 1 (implemented in C)

$$f(t) = \sqrt{\frac{t}{1+t}}\,\mathrm{e}^{-t/2}, \quad t \in (0, \infty)$$

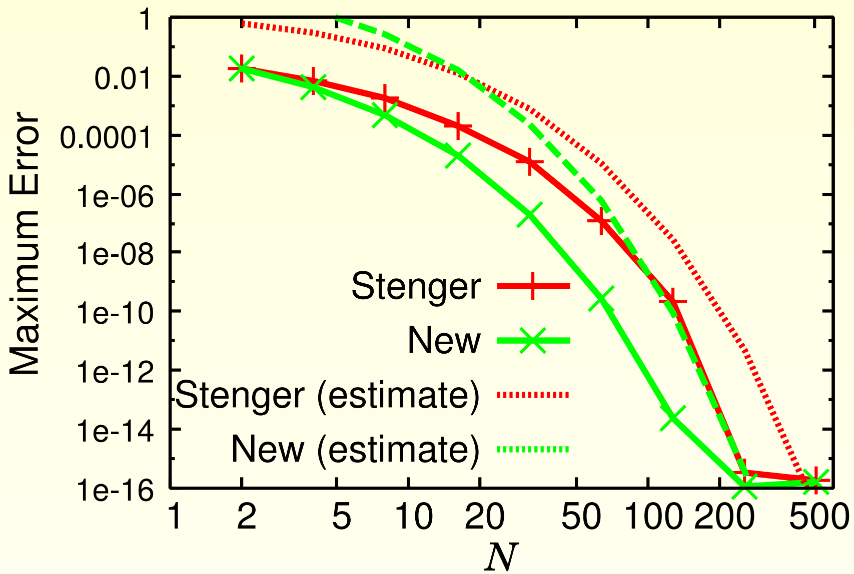


	K	μ	d
$\psi(x)$	1	$1/2$	$\pi/2$
$\phi(x)$	1	$1/2$	3

- maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- "estimate" (dotted line) bounds the actual error (solid line)
- New conformal map $\phi(x)$ is better than existing one $\psi(x)$

Numerical example 2 (implemented in C)

$$f(t) = t^{\pi/4}\,\mathrm{e}^{-t}, \quad t \in (0, \infty)$$



	K	μ	d
$\psi(x)$	1.63	$3/4$	$\pi/2$
$\phi(x)$	3.77	$7/8$	3

- maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- “estimate” (dotted line) bounds the actual error (solid line)
- New conformal map $\phi(x)$ is better than existing one $\psi(x)$

4 Summary and future work

Summary and future work

Summary

Improvement of the conformal map for the Sinc approximation

- $t = \psi(x) = \operatorname{arcsinh}(\mathrm{e}^x)$ has been used (for the case 3)

  $\overset{\text{New!}}{\rightarrow}$ $t = \phi(x) = \log(1 + \mathrm{e}^x)$ is proposed

- $|\text{Error}| \leq C\sqrt{N}\exp(-\sqrt{\pi d \mu N}) \quad (0 < d \leq \pi/2)$

  $\overset{\text{New!}}{\rightarrow}$ $|\text{Error}| \leq C'\sqrt{N}\exp(-\sqrt{\pi d \mu N}) \quad (0 < d < \pi)$

Future work

- Any better conformal map?

ToDo: Bound the transformed function on $\mathscr{D}_d$

Case 2 $$f(t) = \left(\frac{t}{1+t^2}\right)^{\mu} g(t) = \left(\frac{\psi(x)}{1+\{\psi(x)\}^2}\right)^{\mu} g(\psi(x))$$

$(\psi(x) = \psi_{\mathsf{SE2}}(x)$ or $\psi_{\mathsf{DE2}}(x))$

- $|g(\psi(z))| \leq K$: Suppose given by users (depends on the problem).
- $\left|\frac{\psi(z)}{1+\{\psi(z)\}^2}\right| \leq ??$: Evaluated by this study (always the same)

Existing bound

$$\left|\frac{\psi_{\mathsf{SE2}}(z)}{1+\{\psi_{\mathsf{SE2}}(z)\}^2}\right| \leq C\,\mathrm{e}^{-|\operatorname{Re} z|}, \quad \left|\frac{\psi_{\mathsf{DE2}}(z)}{1+\{\psi_{\mathsf{DE2}}(z)\}^2}\right| \leq C\,\mathrm{e}^{-\pi\exp(|\operatorname{Re} z|)/2}$$

The explicit form of C's is revealed in this study (for Case 1–3), which enables us to obtain the desired explicit error bound.