New conformal map for the Sinc approximation for exponentially-decaying functions over the semi-infinite interval

Tomoaki OKAYAMA (Hiroshima City Univ.)
Joint work with: Y. SHINTAKU and E. KATSUURA

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1 Sinc approximation combined with conformal map
(Introduction & summary)
Sinc approximation on \( \mathbb{R} \)

\[
F(x) \approx \sum_{j=-N}^{N} F(jh) \text{Sinc}(x/h - j)
\]

\[
\text{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}
\]

Exponential convergence \( O(\exp(-c\sqrt{N})) \) can be attained if:

- \( F \) is defined on the whole real line \( (x \in \mathbb{R}) \)
- \( |F(x)| \) decays quickly (exponentially) as \( x \to \pm \infty \)

**Question** What should a user do if \( F \) does not satisfy them?

\( \rightarrow \) Employ a suitable variable transformation
Four typical cases that Stenger [1] considered

Let $g$ be a smooth and bounded function and $\mu > 0$

1. $I_1 = (-\infty, \infty)$ \hspace{1cm} $f(t) = \left(\frac{1}{1 + t^2}\right)^\mu g(t)$ \hspace{1cm} polynomial decay

2. $I_2 = (0, \infty)$ \hspace{1cm} $f(t) = \left(\frac{t}{1 + t^2}\right)^\mu g(t)$ \hspace{1cm} polynomial decay

3. $I_3 = (0, \infty)$ \hspace{1cm} $f(t) = \left(\frac{t}{1 + t}\right)^\mu e^{-\mu t} g(t)$ \hspace{1cm} exponential decay

4. $I_4 = (a, b)$ \hspace{1cm} $f(t) = \{(t - a)(b - t)\}^\mu g(t)$ \hspace{1cm} polynomial decay
Conformal maps $\psi$ for the 4 cases

Variable transformation $t = \psi(x)$ (Stenger [1])

$I_1 = (-\infty, \infty)$

$$t = \psi(x) = \sinh x$$

$I_2 = (0, \infty)$

$$t = \psi(x) = e^x$$

$I_3 = (0, \infty)$

$$t = \psi(x) = \text{arcsinh}(e^x)$$

$I_4 = (a, b)$

$$t = \psi(x) = \frac{b-a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}$$

$$f(t) = \left(\frac{1}{1+t^2}\right)^\mu g(t)$$

$$f(t) = \left(\frac{t}{1+t^2}\right)^\mu g(t)$$

$$f(t) = \left(\frac{t}{1+t}\right)^\mu e^{-\mu t} g(t)$$

$$f(t) = \left\{(t-a)(b-t)\right\}^\mu g(t)$$
Conformal maps $\psi$ for the 4 cases

Variable transformation $t = \psi(x)$ (Stenger [1])

$I_1 = (\infty, \infty)$
$$t = \psi(x) = \sinh x$$

$I_2 = (0, \infty)$
$$t = \psi(x) = e^x$$

$I_3 = (0, \infty)$
$$t = \psi(x) = \text{arcsinh}(e^x)$$

$I_4 = (a, b)$
$$t = \psi(x) = \frac{b-a}{2} \tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}$$

$f(t) = \left(\frac{1}{1 + t^2}\right)^\mu g(t)$

$f(\psi(x)) = \left(\frac{e^x}{1 + e^{2x}}\right)^\mu g(e^x)$

$f(t) = \left(\frac{t}{1 + t}\right)^\mu e^{-\mu t} g(t)$

$f(t) = \left\{(t - a)(b - t)\right\}^\mu g(t)$
1 Sinc approximation combined with conformal map (Intro & summary)

Numerical example (implemented in C)

Case 3 \[ f(t) = \sqrt{\frac{t}{1 + t^2}} e^{-t/2}, \quad t \in (0, \infty) \]

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- Maximum error on \( t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50} \) is shown
- Exponential convergence \( O(\exp(-\frac{\pi}{2} \sqrt{N})) \) is obtained

\[ t = \psi(x) = \text{arcsinh}(e^x) \]
Contribution 1: improve the conformal map $\psi$

**Case 3**

$$f(t) = \sqrt{\frac{t}{1 + t^2}} e^{-t/2}, \quad t \in (0, \infty)$$

- **Stenger** $t = \psi(x) = \text{arcsinh}(e^x)$
- **New** $t = \phi(x) = \log(1 + e^x)$

- Maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown
- Faster convergence: $O(\exp(-\frac{\pi}{2}\sqrt{N})) \rightarrow O(\exp(-\frac{\pi}{\sqrt{2}}\sqrt{N}))$
Sinc approximation combined with conformal map (Intro & summary)

Error bound has been given in the case of $\psi$

Case 3: $f(t) = \sqrt{\frac{t}{1 + t^2}} e^{-t/2}$, $t \in (0, \infty)$

Stenger: $t = \psi(x) = \text{arcsinh}(e^x)$

$|\text{Error}| \leq 4.662\sqrt{N} \exp\left(-\frac{\pi}{2}\sqrt{N}\right)$

New: $t = \phi(x) = \log(1 + e^x)$

- Maximum error on $t = 2^{-50}$, $2^{-49.5}$, $2^{-49}$, $\ldots$, $2^{49.5}$, $2^{50}$ is shown
- Faster convergence: $O(\exp\left(-\frac{\pi}{2}\sqrt{N}\right)) \to O(\exp\left(-\frac{\pi}{\sqrt{2}}\sqrt{N}\right))$
Contribution 2: Error bound is given for $\phi$

Case 3

$$f(t) = \sqrt{\frac{t}{1 + t^2}} e^{-t/2}, \quad t \in (0, \infty)$$

- **Stenger**
  $$t = \psi(x) = \text{arcsinh}(e^x)$$
  $$|\text{Error}| \leq 4.662\sqrt{N} \exp\left(-\frac{\pi}{2}\sqrt{N}\right)$$

- **New**
  $$t = \phi(x) = \log(1 + e^x)$$
  $$|\text{Error}| \leq 14.56\sqrt{N} \exp\left(-\frac{\pi}{\sqrt{2}}\sqrt{N}\right)$$

- Maximum error on $t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50}$ is shown

- Faster convergence: $O(\exp(-\frac{\pi}{2}\sqrt{N})) \rightarrow O(\exp(-\frac{\pi}{\sqrt{2}}\sqrt{N}))$
2 Improvement of the conformal map and giving the error bound
Definition: the strip domain $\mathcal{D}_d$

The transformed function $F(x) = f(\psi(x))$ should be analytic on the complex domain $\mathcal{D}_d$:

$$\mathcal{D}_d = \{ z \in \mathbb{C} : |\text{Im} \ z| < d \}$$

$$(\Rightarrow |\text{Error}| = O(\exp(-\pi d/h)))$$

Example $F(x) = \frac{1}{1 + x^2}$ is analytic on $\mathcal{D}_d$ with $d = 1$
2 Improvement of the conformal map and giving the error bound

Idea of the improvement

\[ F'(x) = f'(\psi(x))\psi'(x) \]

Existing conformal map \( \psi \)

\[ \psi'(x) = \frac{1}{\sqrt{1 + e^{-2x}}} \]

(analytic on \( \mathcal{D}_d \) with \( d = \pi/2 \))

New conformal map \( \phi \)

\[ \phi'(x) = \frac{1}{1 + e^{-x}} \]

(analytic on \( \mathcal{D}_d \) with \( d = \pi \))

⇒ Higher convergence rate is expected (cf. \( O(\exp(-\pi d/h)) \))
Improvement of the conformal map and giving the error bound

**Error bound in the case of \( \psi \) (Existing result)**

Existing conformal map: \( \psi(x) = \text{arcsinh}(e^x) \)

**Theorem** (Okayama, 2018)

- \( f(\psi(\cdot)) \) is analytic on \( D_d \). (\( 0 < d \leq \pi/2 \))

- \( \forall z \in \psi(D_d), \ |f(z)| \leq K \left| \frac{z}{1+z} \right|^{\mu} |e^{-z}|^{\mu}. \)

\[ \Rightarrow \sup_{t \in (0, \infty)} |\text{Error}(t)| \leq C \sqrt{N} \exp(-\sqrt{\pi d \mu N}), \]

where

\[ C = \frac{2K}{\sqrt{\pi d \mu}} \left\{ \frac{2 \cdot 2^{\mu}}{\sqrt{\pi d \mu(1 - e^{-2\sqrt{\pi d \mu}}) \cos^{2\mu}(d/2)}} + 1 \right\}. \]
2 Improvement of the conformal map and giving the error bound

Error bound in the case of $\phi$ (New!)

New conformal map: $\phi(x) = \log(1 + e^x)$

**Theorem** (This work)

- $f(\phi(\cdot))$ is analytic on $D_d$. $(0 < d < \pi)$

- $\forall z \in \phi(D_d), |f(z)| \leq K \left| \frac{z}{1+z} \right|^{\mu} |e^{-z}|^{\mu}$.

$\implies$ $\sup_{t \in (0, \infty)} |\text{Error}(t)| \leq C \sqrt{N} \exp(-\sqrt{\pi d\mu N})$,

where

$$C = \frac{2K}{\sqrt{\pi d\mu}} \left\{ \frac{2 \cdot \{e / (e - 1)\}^{\mu/2}}{\sqrt{\pi d\mu} (1 - e^{-2\sqrt{\pi d\mu}}) \cos^{2\mu} (d/2)} + 1 \right\}.$$
3 Numerical examples
Numerical example 1 (implemented in C)

\[ f(t) = \sqrt{\frac{t}{1+t}} e^{-t/2}, \quad t \in (0, \infty) \]

- maximum error on \( t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50} \) is shown
- “estimate” (dotted line) bounds the actual error (solid line)
- New conformal map \( \phi(x) \) is better than existing one \( \psi(x) \)

\[ \begin{array}{cccc} 
\psi(x) & K & \mu & d \\
\phi(x) & 1 & 1/2 & \pi/2 \\
\end{array} \]
Numerical example 2 (implemented in C)

\[ f(t) = t^{\pi/4} e^{-t}, \quad t \in (0, \infty) \]

- maximum error on \( t = 2^{-50}, 2^{-49.5}, 2^{-49}, \ldots, 2^{49.5}, 2^{50} \) is shown
- “estimate” (dotted line) bounds the actual error (solid line)
- New conformal map \( \phi(x) \) is better than existing one \( \psi(x) \)
4 Summary and future work
Summary

Improvement of the conformal map for the Sinc approximation

- \( t = \psi(x) = \text{arcsinh}(e^x) \) has been used (for the case 3)

New! \( t = \phi(x) = \log(1 + e^x) \) is proposed

- \( |\text{Error}| \leq C\sqrt{N} \exp(-\sqrt{\pi d \mu N}) \quad (0 < d \leq \pi/2) \)

New! \( |\text{Error}| \leq C'\sqrt{N} \exp(-\sqrt{\pi d \mu N}) \quad (0 < d < \pi) \)

Future work

- Any better conformal map?
4 Summary and future work

**ToDo: Bound the transformed function on** $\mathcal{D}_d$

**Case 2**

$$f(t) = \left( \frac{t}{1 + t^2} \right)^\mu g(t) = \left( \frac{\psi(x)}{1 + \{\psi(x)\}^2} \right)^\mu g(\psi(x))$$

$$\psi(x) = \psi_{SE2}(x) \text{ or } \psi_{DE2}(x)$$

- $|g(\psi(z))| \leq K$: Suppose given by users (depends on the problem).
- $\left| \frac{\psi(z)}{1 + \{\psi(z)\}^2} \right| \leq ??$: Evaluated by this study (always the same)

**Existing bound**

$$\left| \frac{\psi_{SE2}(z)}{1 + \{\psi_{SE2}(z)\}^2} \right| \leq C e^{-|\text{Re} z|}, \quad \left| \frac{\psi_{DE2}(z)}{1 + \{\psi_{DE2}(z)\}^2} \right| \leq C e^{-\pi \exp(|\text{Re} z|)/2}$$

The explicit form of $C$’s is revealed in this study (for Case 1–3), which enables us to obtain the desired explicit error bound.