Absolute value circulant preconditioners for nonsymmetric Toeplitz-related systems

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Introduction

- We are interested in solving the systems defined by analytic functions $h(z)$ of Toeplitz matrices $T_n$ using Krylov subspace methods, i.e.

$$h(T_n)\mathbf{x} = \mathbf{b}.$$ 

- Examples: $h(T_n) = e^{T_n}$, $\sin T_n$, $\cos T_n$, etc.

Motivations:

- 1. Such a problem has been recently of interest for example in [Jin, Zhao, and Tam, 2014] and [Bai, Jin, and Yao, 2015].

  - Applications: $e^{T_n}$ arises in option pricing, etc.

- 2. The idea of converting a nonsymmetric problem to a symmetric one + absolute value preconditioning.
Toepelliz matrices

\[
T_n = \begin{bmatrix}
a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1} \\
a_1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
a_{n-2} & \ddots & \ddots & \ddots & a_{-1} \\
a_{n-1} & a_{n-2} & \cdots & a_1 & a_0
\end{bmatrix}
\]

For all \( f \in L^1[-\pi, \pi] \), we let

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, dx, \quad k = 0, \pm 1, \pm 2, \ldots,
\]

be the Fourier coefficients of \( f \). The function \( f \) is called the generating function/spectral symbol of \( T_n \).
Preconditioning for $A_n$

- Let $A_n \in \mathbb{C}^{n \times n}$.

\[ A_n x = b \]

- (Left) Preconditioning:

\[ P_n^{-1} A_n x = P_n^{-1} b \]

- To speed up a typical Krylov subspace method, a good preconditioner $P_n$ should satisfy:
  - 1. The product $P_n^{-1} d$ for any vector $d$ needs to be efficiently computed.
  - 2. The spectrum of $P_n^{-1} A_n$ should be clustered.
Circulant matrices

\[
C_n = \begin{bmatrix}
  c_0 & c_{n-1} & \cdots & c_2 & c_1 \\
  c_1 & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  c_{n-2} & \ddots & \ddots & \ddots & c_{n-1} \\
  c_{n-1} & c_{n-2} & \cdots & c_1 & c_0
\end{bmatrix}
\]

\(\triangleright\) (Eigendecomposition [Davis, 1979]) For any \(C_n \in \mathbb{C}^{n \times n}\),

\[
C_n = F_n^* \Lambda_n F_n
\]

where \(F_n\) is the Fourier matrix.
Example: Optimal circulant preconditioners for $T_n$

- We let

$$M_{F_n} = \{ F_n^* \Lambda_n F_n \mid \Lambda_n \text{ is any } n\text{-by-}n \text{ diagonal matrix} \}$$

be the set of all circulant matrices.

- The optimal preconditioner $c(T_n) \in \mathbb{C}^{n\times n}$ [T. Chan, 1988] for $T_n$ is defined to be the minimiser of

$$\| T_n - C_n \|_F$$

over all $C_n \in M_{F_n}$, where $\| \cdot \|_F$ is the Frobenius norm.
1. Symmetrising real $T_n$

Instead of directly dealing with a real Toeplitz system $T_n \mathbf{x} = b$, [Pestana and Wathen, 2015] suggested that one can premultiply it by the anti-identity matrix $Y_n$, defined as

$$Y_n = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & \ldots & 1 \end{bmatrix} \in \mathbb{R}^{n \times n},$$

to obtain the symmetric system $Y_n T_n \mathbf{x} = Y_n b$. 
1. Symmetrising real $T_n$

Since $Y_n T_n$ is symmetric yet (possibly) indefinite, we need a symmetric positive definite preconditioner to use MINRES.
2. Absolute value circulant preconditioner $|C_n|$

- The authors also proposed the absolute value circulant preconditioner
  
  $$|C_n| = F_n^* \Lambda_n F_n.$$  

- $|C_n|$ is symmetric positive definite provided that $C_n$ is nonsingular.

- They further showed that the eigenvalues of the preconditioned matrix $|C_n|^{-1} Y_n T_n$ are clustered around $\pm 1$.

- One can then use for example MINRES for $Y_n T_n$ with guaranteed convergence that depends only on its spectrum.
Example

For example, we consider the $n \times n$ Grcar matrix $T_n$, i.e.

$$T_n = \begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots & 1 \\
-1 & 1 & & \\
& -1 & 1 & \\
& & & -1 & 1
\end{bmatrix} \in \mathbb{R}^{n \times n}.$$
Example

We have the corresponding symmetrised (Hankel) matrix

$$Y_n T_n = \begin{bmatrix}
-1 & 1 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & \ddots & \ddots & \ddots & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix} \in \mathbb{R}^{n \times n}.$$
Example

For example, $|c(T_4)|$ derived from $T_4$ is given by

$$
|c(T_4)| = \begin{bmatrix}
1.4231 & 0.1250 & 0.0769 & 0.1250 \\
0.1250 & 1.4231 & 0.1250 & 0.0769 \\
0.0769 & 0.1250 & 1.4231 & 0.1250 \\
0.1250 & 0.0769 & 0.1250 & 1.4231
\end{bmatrix}.
$$
Problem setup

▶ Under certain assumptions, we show that $|h(c(T_n))|$ is an effective preconditioner for

$$h(T_n)x = b.$$ 

▶ $T_n$ is the Toeplitz matrix generated by a function $f \in C[-\pi, \pi]$, the Banach space of continuous complex-valued functions defined on $[-\pi, \pi]$ with the supremum norm $||\cdot||_\infty$.

▶ $c(T_n)$: the optimal circulant preconditioner for $T_n$.

▶ Note that $h(T_n)$ is not Toeplitz in general.
1. Symmetrising real $h(T_n)$

- Given a $h(T_n) \in \mathbb{R}^{n \times n}$, we can premultiply it by $Y_n$ to obtain a symmetric matrix $Y_n h(T_n)$.

- As $Y_n h(T_n)$ is (possibly) indefinite, MINRES with a symmetric positive definite preconditioner should be used.
2. $|h(C_n)|$ as a preconditioner for nonsymmetric $h(T_n)$

- $|h(C_n)|$ is a circulant matrix as

$$|h(C_n)| = F_n^* |h(\Lambda_n)| F_n.$$ 

- Hence, the product $|h(C_n)|^{-1} d$ for any vector $d$ can be efficiently computed by a few Fast Fourier Transforms (FFTs) in $O(n \log n)$ operations.
2. $|h(C_n)|$ as a preconditioner for nonsymmetric $h(T_n)$

**Theorem (Hon, 2018 & Hon and Wathen 2018)**

Suppose $h(z)$ with the radius of convergence $r$ is analytic on $|z| < r$. Let $f \in C[-\pi, \pi]$ with real Fourier coefficients such that $2\|f\|_{\infty} < r$. Let $T_n \in \mathbb{R}^{n \times n}$ be the Toeplitz matrix generated by $f$, let $c(T_n) \in \mathbb{R}^{n \times n}$ be the optimal circulant preconditioner for $T_n$, and let $Y_n \in \mathbb{R}^{n \times n}$ be the anti-identity matrix. If $\|h(c(T_n))^{-1}\|_2$ is uniformly bounded w.r.t. $n$, then for all $\epsilon > 0$ there exist positive integers $N_\epsilon$ and $M_\epsilon$ such that for all $n > N_\epsilon$

$$|h(c(T_n)))^{-1} Y_n h(T_n) = Q_n + R_n + E_n,$$

where $Q_n$ is symmetric and orthogonal,

$$\text{rank}(R_n) \leq M_\epsilon,$$

and

$$\|E_n\|_2 \leq \epsilon.$$
Example. Nonsymmetric sinh $T_n$

- Consider sinh $T_n$, where $T_n$ is the Grcar matrix.
- For sufficiently large $n$, we have
  \[ |\sinh c(T_n)|^{-1} Y_n \sinh T_n = Q_n + \tilde{R}_n + \tilde{E}_n. \]
- Since $Y_n \sinh T_n$, $|\sinh c(T_n)|$, and $Q_n$ are all symmetric, the eigenvalues of
  \[ |\sinh c(T_n)|^{-1} Y_n \sinh T_n \]
  are mostly close to ±1.
Example

For example, when $n = 4$,

$$\sinh T_4 = \begin{bmatrix}
0.8351 & 0.9110 & 0.7142 & 2.1095 \\
-1.1614 & 0.5847 & 0.2134 & 0.7142 \\
0.4740 & -0.9378 & 0.5847 & 0.9110 \\
-0.2236 & 0.4740 & -1.1614 & 0.8351
\end{bmatrix} \in \mathbb{R}^{4 \times 4}.$$
Example

We have the symmetrised matrix

\[
Y_4 \sinh T_4 = \begin{bmatrix}
-0.2236 & 0.4740 & -1.1614 & 0.8351 \\
0.4740 & -0.9378 & 0.5847 & 0.9110 \\
-1.1614 & 0.5847 & 0.2134 & 0.7142 \\
0.8351 & 0.9110 & 0.7142 & 2.1095
\end{bmatrix}.
\]
Example

The corresponding $|\sinh c(T_4)|$ derived from $T_4$ is

$$
|\sinh c(T_4)| = \begin{bmatrix}
1.6394 & 0.2971 & 0.5568 & 0.2971 \\
0.2971 & 1.6394 & 0.2971 & 0.5568 \\
0.5568 & 0.2971 & 1.6394 & 0.2971 \\
0.2971 & 0.5568 & 0.2971 & 1.6394
\end{bmatrix}.
$$
Example. Nonsymmetric sinh $T_n$

- sinh $T_n$, where $T_n$ is a Grcar matrix.

Table: Numbers of iterations with MINRES for $Y_n \sinh T_n$

| $n$   | with no preconditioner | with $|\sinh c(T_n)|$ |
|-------|------------------------|---------------------|
| 32    | 39                     | 21                  |
| 64    | 106                    | 24                  |
| 128   | 172                    | 22                  |
| 256   | 391                    | 20                  |
Example. Nonsymmetric $\sinh T_n$

- $\sinh T_n$, where $T_n$ is a Grcar matrix.

**Figure**: Spectrum of $Y_n \sinh T_n$ at $n = 256$ (i) without a preconditioner or (ii) with the preconditioner $|\sinh c(T_n)|$. 
Conclusions

- We have provided the followings:
  1. One way ($Y_n$) to convert a nonsymmetric problem to a symmetric one + the absolute value preconditioner $|h(c(T_n))|$, and
  2. a theorem that accounts for the effectiveness of $|h(c(T_n))|$.

- An extension of this work to the multilevel Toeplitz case is straightforward.
Future work

▶ A direction for future work is to design effective preconditioners for ill-conditioned symmetric matrix $Y_n T_n$.

▶ Another is to investigate the asymptotic spectral distribution of symmetric matrix $Y_n T_n$.

- Knowledge about the spectrum of $Y_n T_n$ could help designing good preconditioners.
The following simple example illustrates the point: consider

\[ T_n = \begin{bmatrix} 2 & 1 & 2 & \cdots & \cdots & 1 & 2 \\ 1 & 2 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 2 \end{bmatrix} \in \mathbb{R}^{n \times n}. \]

Its spectral symbol is \( f(x) = 2 + e^{ix} \) and \( n = 512 \).
Asymptotic spectral distribution of symmetrised Toeplitz matrices

Figure: Singular value distribution of $Y_n T_n$ with $T_n$ generated by $f(x) = 2 + e^{ix}$ and $n = 512$.

The singular values of $Y_n T_n$ are distributed as

$$|f(x)| = \sqrt{5 + 4 \cos x}.$$
Asymptotic spectral distribution of symmetrised Toeplitz matrices

Figure: Spectral distribution of $Y_n T_n$ with $T_n$ generated by $f(x) = 2 + e^{ix}$ and $n = 512$.

- The eigenvalues of $Y_n T_n$ seem to be distributed as $\pm |f|$. In fact, roughly half of them are negative/positive. This is joint work with Stefano Serra-Capizzano.
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