Convergence of a Modified Newton Method for a Matrix Polynomial Equation Arising in Stochastic Problems

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Introduction

Matrix Polynomial Equation

$$
\begin{aligned}
P(X) &= \sum_{k=0}^{n} A_k X^k \\
&= A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 \\
&= 0
\end{aligned}
\tag{1.1}
$$

Matrix Polynomial Equation

$$P(X) = \sum_{k=0}^{n} A_k X^k = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 = 0 \tag{1.1}$$

Assumption of the MPE (Generalization of Stochastic Problems)

1. For $k \in \mathbb{N}_0$, $A_k \geq 0$ except A_1,
2. $-A_1$ is a nonsingular M-matrix,
3. For $k \in \mathbb{N}_0$, A_k is irreducible.

Newton's Method

Let an equation

$$f(x) = 0$$

be given where f is a real function. Then, for given x_0, we can define a sequence $\{x_i\}_{i=0}^{\infty}$ by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

If $f'(x_i) \neq 0$ for all $i \geq 0$ and $\{x_i\}$ converges, then the limit is a solution of $f(x) = 0$.

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We can apply this idea to nonlinear matrix equations with the Fréchet derivative.

Preliminaries

Nonnegative Matrices and Partial Order Relations

Definition

Let $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and $B = [b_{ij}] \in \mathbb{R}^{m \times n}$.

1. A : a positive matrix (nonnegative matrix) if

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2. We say that $A > B$ ($A \geq B$) if $A - B$ is positive(nonnegative).
3. A and B are comparable if $A \geq B$ or $A \leq B$.

M-matrices

Definition

Let $A \in \mathbb{R}^{m \times m}$. A is a Z-matrix if all its off-diagonal elements are nonpositive.

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A matrix $A \in \mathbb{R}^{m \times m}$ is an M-matrix if $A = rI - B$ for some $B \geq 0$ with $r \geq \rho(B)$ where ρ is the spectral radius.

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It is clear that any Z-matrix A can be written as $sI - B$ with $B \geq 0$ and $s \in \mathbb{R}$. Then, an M-matrix also is a Z-matrix.

M-matrices as Inverse-Positive matrices

Theorem

For a Z-matrix A, the following are equivalent:

1. A is a nonsingular M-matrix.
2. A^{-1} is nonnegative.
3. $Av > 0$ for some vector $v > 0$.
4. All eigenvalues of A have positive real parts.

Kronecker Product and Vec Operator

For $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$, the Kronecker Product of A and B is the matrix $(A \otimes B) \in \mathbb{C}^{mp\times nq}$ defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}, \text{ where } A = [a_{ij}], B = [b_{ij}].$$

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The Vec Operator of A is the column vector

$$\mathrm{vec}(A) = \begin{bmatrix} \mathbf{a}_1^T & \mathbf{a}_2^T & \cdots & \mathbf{a}_n^T \end{bmatrix}^T, \text{ where } A = \left[\ \mathbf{a}_1 \mid \mathbf{a}_2 \mid \cdots \mid \mathbf{a}_n\ \right].$$

Solving a Linear Matrix Equation

Lemma

Let $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{p\times q}$, and $C \in \mathbb{C}^{m\times q}$ be given and let $X \in \mathbb{C}^{n\times p}$ be unknown. Then, the linear matrix equation

$$AXB = C$$

is equivalent to the system of qm equations in np unknowns given by

$$(B^T \otimes A)\mathrm{vec}(X) = \mathrm{vec}(C),$$

that is, $\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$.

Newton's Method for Matrix Polynomial Equations

Sufficient Condition of the Existence of the Solution

Theorem

Let the MPE (1.1) with 1), 2) in Assumption be given. Then, there exists the minimal nonnegative solution if

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Theorem

Let the MPE (1.1) with 1), 2) in Assumption be given. Then, there exists the minimal nonnegative solution if

$$B = -\sum_{k=0}^{n} A_k \tag{3.1}$$

is a nonsingular or singular irreducible M-matrix.

Newton's Method for the MPE

For the equation (1.1), each step of the Newton iteration with a given X_0 can be written as

$$X_{i+1} = X_i - P_{X_i}'^{-1}(P(X_i)), \tag{3.2}$$

where $P_X'(H) = \sum_{k=1}^{n}\sum_{l=0}^{k-1} A_k X^l H X^{k-l-1}$ is the Fréchet derivative of P at X to H.

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(3.2) can be separated into two parts as

$$\begin{cases} P'_{X_i}(H_i) = -P(X_i), \\ X_{i+1} = X_i + H_i. \end{cases} \tag{3.3}$$

Newton's Method

The general approach for solving (3.3) is to solve the $m^2 \times m^2$ linear system such as

$$\mathcal{P}'_{X_i}\mathrm{vec}(H_i) = \mathrm{vec}(-P(X_i)),$$

where

$$\begin{aligned}\mathcal{P}'_{X_i} &= \mathrm{vec} \circ P'_{X_i} \circ \mathrm{vec}^{-1} \\ &= \sum_{k=1}^{n}\sum_{l=0}^{k-1}(X^{k-l-1})^T \otimes A_k X^l.\end{aligned}$$

Convergence of the Newton Sequence

Theorem

Suppose that the MPE (1.1) with Assumption of the MPE and (3.1) is a nonsingular or singular irreducible M-matrix. Then, the Newton sequence $\{X_i\}$ with $X_0 = 0$ is well-defined, is monotone nondecreasing, and converges to the elementwise minimal positive solution S. Furthermore, $-\mathcal{P}'_{X_i}$ is a nonsingular irreducible M-matrix for $i \geq 1$, and $-\mathcal{P}'_S$ is an irreducible M-matrix.

Theorem

If the matrix $-\mathcal{P}'_S$ is nonsingular, then for $X_0 = 0$, the Newton sequence $\{X_i\}$ converges to S, quadratically.

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For a nonzero $N_0 \in \mathcal{N}$, define $\mathcal{B}_{N_0} : \mathcal{N} \to \mathcal{N}$ given by

$$\mathcal{B}_{N_0}(N) = \mathbf{P}_{\mathcal{N}} P_S''(N_0, N), \text{ where}$$

$$P_X''(K,H) = \sum_{k=2}^{n}\sum_{l=0}^{k-2}\sum_{j=0}^{l} A_k\left(X^l H X^j K X^{n-l-j-2} + X^l K X^j H X^{n-l-j-2}\right)$$

is the second Fréchet derivative at X.

Theorem

Let $\mathcal{B}_{N_0}$ be invertible for some nonzero $N_0 \in \mathcal{N}$, $\mathcal{N} = \mathrm{span}\{N_0\} \oplus \mathcal{N}_1$ for some subspace $\mathcal{N}_1$, and let

$$W(\rho,\theta,\eta) = \left\{ X \;\middle|\; \begin{array}{l} 0 < \|X-S\| < \rho, \\ \|\mathbf{P}_{\mathcal{M}}(X-S)\| \le \theta \|\mathbf{P}_{\mathcal{N}}(X-S)\|, \\ \|(\mathbf{P}_{\mathcal{N}} - \mathbf{P}_0)(X-S)\| \le \eta \|\mathbf{P}_{\mathcal{N}}(X-S)\| \end{array} \right\}$$

where $\mathbf{P}_0$ is the projection onto $\mathrm{span}\{N_0\}$ parallel to $\mathcal{N}_1 \oplus \mathcal{M}$. If $X_0 \in W(\rho_0,\theta_0,\eta_0)$ for ρ_0,θ_0,η_0 sufficiently small, then the Newton sequence $\{X_i\}$ is well-defined and $\|P'^{-1}_{X_i}\| \le c\|X_i - S\|^{-1}$ for all $i \ge 1$ and some constant $c > 0$. Moreover,

$$\lim_{i\to\infty} \frac{\|X_{i+1}-S\|}{\|X_i - S\|} = \frac{1}{2}, \quad \lim_{i\to\infty} \frac{\|\mathbf{P}_{\mathcal{M}}(X_i - S)\|}{\|\mathbf{P}_{\mathcal{N}}(X_i - S)\|^2} = 0.$$

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So, the MPE satisfies the previous theorem.

Convergence rate of Newton's method for the MPE

Theorem

If $\mathcal{P}'_S$ is a singular M-matrix and the convergence rate of the Newton sequence $\{X_i\}$ of the MPE (1.1) is not quadratic, then $\|P'^{-1}_{X_i}\| \leq c\|X_i - S\|^{-1}$ for all $i \geq 1$ and some constant $c > 0$. Moreover,

$$\lim_{i\to\infty} \frac{\|X_{i+1} - S\|}{\|X_i - S\|} = \frac{1}{2}, \quad \lim_{i\to\infty} \frac{\|\mathbf{P}_{\mathcal{M}}(X_i - S)\|}{\|\mathbf{P}_{\mathcal{N}}(X_i - S)\|^2} = 0.$$

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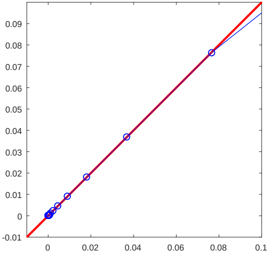

So, we can guess that the Newton iteration can be reduced if we choose suitable λ_i, where the modified Newton iteration is defined by

$$X_{i+1} = X_i - \lambda_i F_{X_i}'^{-1}(F(X_i)). \tag{5.1}$$

The Modified Newton Method for MPEs

Theorem

Let $Y_{i+1} = X_i - 2P'^{-1}_{X_i}(P(X_i)), p = \|\mathbf{P}_\mathcal{N}\|$, and let $\varepsilon \in (0, t)$ be given where t is the real root of $f(x) = px^3 + 2px^2 + (9p + 1)x - 1$ in $(0, 1)$. Suppose that $i \geq i_0$ implies that

$$\left| \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} - 2 \right| < \varepsilon, \quad \|\mathbf{P}_\mathcal{M}(\tilde{X}_i)\| < \varepsilon\|\mathbf{P}_\mathcal{N}(\tilde{X}_i)\| \tag{5.2}$$

for some $i_0 \in \mathbb{N}_0$. Then,

$$\|Y_{i+1} - S\| < \|X_{i+1} - S\|. \tag{5.3}$$

Proof. From the definition of Y_{i+1}, we get that

$$\begin{aligned} Y_{i+1} - S &= X_i - 2P'^{-1}_{X_i}(P(X_i)) - S \\ &= 2\left(X_i - P'^{-1}_{X_i}(P(X_i))\right) - X_i - S \\ &= 2X_{i+1} - X_i - 2S + S \\ &= 2\tilde{X}_{i+1} - \tilde{X}_i. \end{aligned}$$

So, we will show that $\|2\tilde{X}_{i+1} - \tilde{X}_i\| < \|\tilde{X}_i\|$.

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So, we will show that $\|2\tilde{X}_{i+1} - \tilde{X}_i\| < \|\tilde{X}_i\|$. From the hypothesis, we obtain that

$$(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\| < \|\tilde{X}_i\| < (1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|. \qquad (5.4)$$

$$2 - \varepsilon < \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} < \frac{(1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|},$$
$$\frac{(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{(1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|} < \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} < 2 + \varepsilon. \tag{5.5}$$

$$\begin{aligned} 2-\varepsilon < \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} < \frac{(1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|}, \\ \frac{(1-\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{(1+\varepsilon)\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|} < \frac{\|\tilde{X}_i\|}{\|\tilde{X}_{i+1}\|} < 2+\varepsilon. \end{aligned} \tag{5.5}$$

$$\Downarrow$$

$$\frac{(2-\varepsilon)(1-\varepsilon)}{1+\varepsilon} < \frac{\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_i)\|}{\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\|} = \frac{c_i}{c_{i+1}} < \frac{(2+\varepsilon)(1+\varepsilon)}{1-\varepsilon} \tag{5.6}$$

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$$\Downarrow$$

$$-\frac{-5\varepsilon+\varepsilon^2}{1+\varepsilon}(-c_{i+1}) < 2c_{i+1}-c_i < \frac{5\varepsilon+\varepsilon^2}{1-\varepsilon}(-c_{i+1}). \tag{5.7}$$

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$$|2c_{i+1}-c_i|<\frac{5\varepsilon+\varepsilon^2}{1-\varepsilon}|c_{i+1}|. \tag{5.8}$$

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$$\Downarrow$$

$$\begin{aligned}\|2\tilde{X}_{i+1} - \tilde{X}_i\| &< \frac{9\varepsilon+2\varepsilon^2+\varepsilon^3}{1-\varepsilon}\|\mathbf{P}_{\mathcal{N}}(\tilde{X}_{i+1})\| \\ &\le \frac{9\varepsilon+2\varepsilon^2+\varepsilon^3}{1-\varepsilon}p\|\tilde{X}_{i+1}\|\end{aligned}$$

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Since $\varepsilon \in (0,t)$,

$$\frac{9\varepsilon+2\varepsilon^2+\varepsilon^3}{1-\varepsilon} < \frac{1}{p} \Longrightarrow \|Y_{i+1} - S\| < \|\tilde{X}_{i+1}\|. \tag{5.9}$$

Numerical Experiments

Numerical Experiments

Let an MPE (1.1) with degree $n = 6$ be given with the following coefficients

$$\begin{cases} A_k = a_i W, & \text{for } k = 0, 2, 3, 4, 5, \\ A_1 = a_1 W - I_m, & \\ A_6 = W, & \end{cases} \tag{6.1}$$

where

$$W = \frac{1}{6200(m-1)}(\mathbf{1}_{m\times m} - I_m), \text{ and } \begin{cases} a_0 = 4096, & a_1 = 56, \\ a_2 = 384, & a_3 = 1312, \\ a_4 = 321, & a_5 = 30. \end{cases} \tag{6.2}$$

Then, A_k's satisfy the Assumption and (3.1) is a singular irreducible M-matrix, i.e., the MPE (1.1) has the minimal nonnegative solution S.

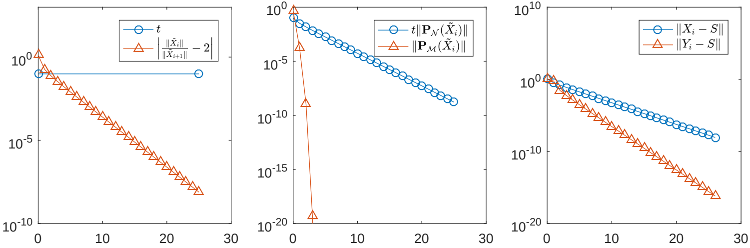
Let $m = 3$. Then, the minimal nonnegative solution

$$S = \begin{bmatrix} \frac{2r+1}{3} & \frac{1-r}{3} & \frac{1-r}{3} \\ \frac{1-r}{3} & \frac{2r+1}{3} & \frac{1-r}{3} \\ \frac{1-r}{3} & \frac{1-r}{3} & \frac{2r+1}{3} \end{bmatrix}$$

where $r \approx -0.3287191$ which is the nearest real root to 0 of an equation

$x^6 + 30x^5 + 321x^4 + 1312x^3 + 384x^2 + 12456x + 4096 = 0.$

Furthermore, $-\mathcal{P}'_S$ is a singular M-matrix, $p = \|\mathbf{P}_{\mathcal{N}}\| = 1$, and $t \approx 0.097985683$ which is the real root of $x^3 + 2x^2 + 10x - 1$.



This experiment shows that, for given i, Y_i is closer to the solution than X_i if it satisfies the conditions of Theorem of the modified Newton method. Furthermore, we obtain that Y_i's for $i \geq 13$ are closer to the solution than any X_i's. It means that we do not need to compute X_i for $i \geq 13$ since Y_{13} is thought to be a numerical minimal nonnegative solution which is close sufficiently to the mathematical minimal nonnegative solution.

Thank you for your attention!