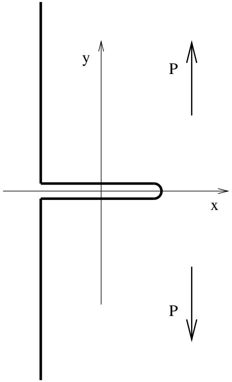
On a Collocation-quadrature Method for the Singular Integral Equation of the Notched Half-plane Problem

P. Junghanns, **Robert Kaiser**

Nasca, 2018

TECHNISCHE UNIVERSITÄT
CHEMNITZ

We consider a notched elastic half plane.



$[-1, \infty) \times \mathbb{R}$	-	elastic medium
P	-	external forces
v	-	crack opening
$(-1, 1)$	-	domain of v

Condition: $v(1) = 0$

The crack problem can be modelled (see Kalandiya, 1975) by the integral equation

$$(\mathcal{S}v + \mathcal{H}_0 v)(x) = -\frac{1+\kappa}{2\mu\mathbf{i}} px + \mathbf{i}\mathcal{C}, \quad -1 < x < 1, \tag{1}$$

where $\mu, \kappa, \mathcal{C} \in \mathbb{R}$ are constants and

$$\begin{aligned}(\mathcal{H}_0 v)(x) &= \frac{1}{\pi\mathbf{i}} \int_{-1}^{1} \left[-\frac{1}{2+y+x} - \frac{2(1+x)}{(2+y+x)^2} + \frac{4(1+x)^2}{(2+y+x)^3} \right] v(y)\, dy \\ &= \frac{1}{\pi\mathbf{i}} \int_{-1}^{1} \mathbf{h}_0 \left(\frac{1+x}{1+y} \right) \frac{v(y)}{1+y}\, dy\end{aligned}$$

with

$$\mathbf{h}_0(t) = -\frac{1}{1+t} - \frac{2t}{(1+t)^2} + \frac{4t^2}{(1+t)^3}$$

as well as

$$(\mathcal{S}v)(x) = \frac{1}{\pi\mathbf{i}} \int_{-1}^{1} \frac{v(y)}{y-x}\, dy.$$

We set

$$\widetilde{\mathbf{L}}_\infty := \bigcap_{1<p<\infty} \mathbf{L}^p(-1,1) \quad \text{and} \quad \mathbf{L}_0 := \bigcup_{1<p<\infty} \mathbf{L}^p(-1,1).$$

Lemma (R.Duduchava, 1978)

Equation (1) *has a unique solution* $v \in \widetilde{\mathbf{L}}_\infty \cap \mathbf{C}^\infty(-1,1)$. *This solution satisfies* $v(1) = 0$ *and is bounded in a neighbourhood of the point* -1.

Assumption: $v' \in \mathbf{L}_0$

We set

$$\widetilde{\mathbf{L}}_\infty := \bigcap_{1<p<\infty} \mathbf{L}^p(-1,1) \quad \text{and} \quad \mathbf{L}_0 := \bigcup_{1<p<\infty} \mathbf{L}^p(-1,1).$$

Lemma (R.Duduchava, 1978)

Equation (1) *has a unique solution* $v \in \widetilde{\mathbf{L}}_\infty \cap \mathbf{C}^\infty(-1,1)$. *This solution satisfies* $v(1) = 0$ *and is bounded in a neighbourhood of the point* -1.

Assumption: $v' \in \mathbf{L}_0$

We set

$$\widetilde{\mathbf{L}}_\infty := \bigcap_{1<p<\infty} \mathbf{L}^p(-1,1) \quad \text{and} \quad \mathbf{L}_0 := \bigcup_{1<p<\infty} \mathbf{L}^p(-1,1).$$

Lemma (R.Duduchava, 1978)

Equation (1) *has a unique solution* $v \in \widetilde{\mathbf{L}}_\infty \cap \mathbf{C}^\infty(-1,1)$. *This solution satisfies* $v(1) = 0$ *and is bounded in a neighbourhood of the point* -1.

Assumption: $v' \in \mathbf{L}_0$

We get

$$\frac{d}{dx}(\mathcal{S}v + \mathcal{H}_0 v)(x) = (\mathcal{S}v' + \widetilde{\mathcal{H}}v')(x),$$

where

$$(\widetilde{\mathcal{H}}v')(x) = \frac{1}{\pi\mathbf{i}} \int_{-1}^{1} \widetilde{\mathbf{h}}\left(\frac{1+x}{1+y}\right) \frac{v'(y)}{1+y}\, dy,$$

and

$$\widetilde{\mathbf{h}}(t) = -\frac{1}{1+t} + \frac{6t}{(1+t)^2} - \frac{4t^2}{(1+t)^3}.$$

Thus we consider the equation

$$(\mathcal{S}\widetilde{u} + \widetilde{\mathcal{H}}\widetilde{u})(x) = -\frac{1+\kappa}{2\mu\mathbf{i}} p =: f, \quad -1 < x < 1. \tag{2}$$

For a Jacobi-weight $v^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta \in (-1,1)$ we introduce the weighted $\mathbf{L}^2$-space

$$\mathbf{L}^2_{\alpha,\beta}, \quad \langle u, w \rangle_{\alpha,\beta} := \int_{-1}^{1} u(x)\overline{w}(x) v^{\alpha,\beta}(x)\, dx.$$

We get

$$\frac{d}{dx}(\mathcal{S}v + \mathcal{H}_0 v)(x) = (\mathcal{S}v' + \widetilde{\mathcal{H}}v')(x),$$

where

$$(\widetilde{\mathcal{H}}v')(x) = \frac{1}{\pi\mathbf{i}}\int_{-1}^{1} \widetilde{\mathbf{h}}\left(\frac{1+x}{1+y}\right)\frac{v'(y)}{1+y}\,dy,$$

and

$$\widetilde{\mathbf{h}}(t) = -\frac{1}{1+t} + \frac{6t}{(1+t)^2} - \frac{4t^2}{(1+t)^3}.$$

Thus we consider the equation

$$(\mathcal{S}\widetilde{u} + \widetilde{\mathcal{H}}\widetilde{u})(x) = -\frac{1+\kappa}{2\mu\mathbf{i}}p =: f, \quad -1 < x < 1. \tag{2}$$

For a Jacobi-weight $v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta \in (-1,1)$ we introduce the weighted $\mathbf{L}^2$-space

$$\mathbf{L}^2_{\alpha,\beta}, \quad \langle u, w\rangle_{\alpha,\beta} := \int_{-1}^{1} u(x)\overline{w}(x)v^{\alpha,\beta}(x)\,dx.$$

Lemma (R.Duduchava, 1978)

If $\alpha \in (0,1)$ and $\beta \in (-1,1)\backslash\{0\}$, then the operator $\mathcal{S} + \widetilde{\mathcal{H}}$ is invertible in $\mathbf{L}^2_{\alpha,\beta}$. If $\widetilde{u}$ is the solution of (2)*, then $\widetilde{u}$ is bounded in a neighbourhood of $x = -1$ and belongs to the space $\mathbf{C}^\infty(-1,1)$. Moreover, $\sqrt{1-x}\,\widetilde{u}(x)$ is locally Hölder continuous in each point of $(-1,1]$ with a Hölder exponent in $(0,1)$ and*

$$v(x) = -\int_x^1 \widetilde{u}(y)\,dy, \quad -1 \leq x \leq 1$$

is the unique solution of (1)*.*

Thus the solution of (2) takes the form

$$\widetilde{u}(x) = \frac{\widetilde{v}(x)}{\sqrt{1-x}}, \quad -1 < x < 1,$$

where $\widetilde{v} : (-1,1] \longrightarrow \mathbb{R}$ is locally Hölder continuous and bounded.

Lemma (R.Duduchava, 1978)

If $\alpha \in (0,1)$ *and* $\beta \in (-1,1)\backslash\{0\}$, *then the operator* $\mathcal{S} + \widetilde{\mathcal{H}}$ *is invertible in* $\mathbf{L}^2_{\alpha,\beta}$. *If* $\widetilde{u}$ *is the solution of* (2), *then* $\widetilde{u}$ *is bounded in a neighbourhood of* $x = -1$ *and belongs to the space* $\mathbf{C}^\infty(-1,1)$. *Moreover,* $\sqrt{1-x}\,\widetilde{u}(x)$ *is locally Hölder continuous in each point of* $(-1,1]$ *with a Hölder exponent in* $(0,1)$ *and*

$$v(x) = -\int_x^1 \widetilde{u}(y)\,dy, \quad -1 \leq x \leq 1$$

is the unique solution of (1).

Thus the solution of (2) takes the form

$$\widetilde{u}(x) = \frac{\widetilde{v}(x)}{\sqrt{1-x}}, \quad -1 < x < 1,$$

where $\widetilde{v} : (-1,1] \longrightarrow \mathbb{R}$ is locally Hölder continuous and bounded.

We are interested in approximate solutions of the form

$$\frac{p_n(x)}{\sqrt{1-x}} = \upsilon^{-\frac{1}{2},0}(x)p_n(x), \quad p_n \in \mathbb{P}_n,$$

where $\mathbb{P}_n$ is the set of all algebraic polynomials of degree less than n.

Since the approximate solution is not of the form $\upsilon^{\alpha,\beta}p_n$ with $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$, we use the isometrically isomorphism

$$\mathcal{J} : \mathbf{L}^2_{\frac{1}{2},\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}, \quad f \mapsto \upsilon^{0,\frac{1}{2}} f$$

and transform $\mathcal{S}\widetilde{u} + \widetilde{\mathcal{H}}\widetilde{u} = f$ in

$$\mathcal{A}u = \mathcal{J}f$$

with $\mathcal{A} = \mathcal{J}(\mathcal{S} + \widetilde{\mathcal{H}})\mathcal{J}^{-1}$ and $u = \mathcal{J}\widetilde{u}$.

We are interested in approximate solutions of the form

$$\frac{p_n(x)}{\sqrt{1-x}} = v^{-\frac{1}{2},0}(x)p_n(x), \quad p_n \in \mathbb{P}_n,$$

where $\mathbb{P}_n$ is the set of all algebraic polynomials of degree less than n.

Since the approximate solution is not of the form $v^{\alpha,\beta}p_n$ with $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$, we use the isometrically isomorphism

$$\mathcal{J} : \mathbf{L}^2_{\frac{1}{2},\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}, \quad f \mapsto v^{0,\frac{1}{2}} f$$

and transform $\mathcal{S}\widetilde{u} + \widetilde{\mathcal{H}}\widetilde{u} = f$ in

$$\mathcal{A}u = \mathcal{J}f$$

with $\mathcal{A} = \mathcal{J}(\mathcal{S} + \widetilde{\mathcal{H}})\mathcal{J}^{-1}$ and $u = \mathcal{J}\widetilde{u}$.

The kernel function of the operator $\mathcal{A}$ is equal to

$$\sqrt{\frac{1+x}{1+y}}\left[\frac{1}{y-x}+\widetilde{\mathbf{h}}\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\right].$$

We have

$$\sqrt{\frac{1+x}{1+y}}\frac{1}{y-x}=\frac{1}{y-x}-\frac{1}{1+\sqrt{\frac{1+x}{1+y}}}\frac{1}{1+y}.$$

Thus $\mathcal{A}=\mathcal{S}+\mathcal{H}$ with

$$(\mathcal{H}u)(x)=\frac{1}{\pi}\int_{-1}^{1}\mathbf{h}\left(\frac{1+x}{1+y}\right)\frac{u(y)}{1+y}\,dy$$

and

$$\mathbf{h}(t)=\sqrt{t}\left(-\frac{1}{1+t}+\frac{6t}{(1+t)^2}-\frac{4t^2}{(1+t)^3}\right)-\frac{1}{1+\sqrt{t}}$$

$$=: -\mathbf{h}_1(t)+6\mathbf{h}_2(t)-4\mathbf{h}_3(t)-\mathbf{h}_0(t).$$

The kernel function of the operator $\mathcal{A}$ is equal to

$$\sqrt{\frac{1+x}{1+y}}\left[\frac{1}{y-x}+\widetilde{\mathbf{h}}\left(\frac{1+x}{1+y}\right)\frac{1}{1+y}\right].$$

We have

$$\sqrt{\frac{1+x}{1+y}}\frac{1}{y-x}=\frac{1}{y-x}-\frac{1}{1+\sqrt{\frac{1+x}{1+y}}}\frac{1}{1+y}.$$

Thus $\mathcal{A}=\mathcal{S}+\mathcal{H}$ with

$$(\mathcal{H}u)(x)=\frac{1}{\pi}\int_{-1}^{1}\mathbf{h}\left(\frac{1+x}{1+y}\right)\frac{u(y)}{1+y}\,dy$$

and

$$\mathbf{h}(t)=\sqrt{t}\left(-\frac{1}{1+t}+\frac{6t}{(1+t)^2}-\frac{4t^2}{(1+t)^3}\right)-\frac{1}{1+\sqrt{t}}$$

$$=: -\mathbf{h}_1(t)+6\mathbf{h}_2(t)-4\mathbf{h}_3(t)-\mathbf{h}_0(t).$$

Let $n \in \mathbb{N}$. We denote by $x_{kn}^\sigma = \cos \frac{2k-1}{2n}\pi$ the Chebychev nodes of first kind and by $\mathcal{L}_n^\sigma$ the Lagrange interpolation operator w.r.t. the nodes x_{kn}^σ. Now we are able to define the quadrature operators $\mathcal{H}_n^s$ by

$$(\mathcal{H}_n^s u)(x) = \frac{1}{\pi}\int_{-1}^{1} \mathcal{L}_n^\sigma \left[\mathbf{h}_s\left(\frac{1+x}{1+\cdot}\right)\frac{v^{\frac{1}{2},\frac{1}{2}}u}{1+\cdot}\right](y)v^{-\frac{1}{2},-\frac{1}{2}}(y)\,dy$$

$$= \frac{1}{n}\sum_{k=1}^{n} v^{\frac{1}{2},\frac{1}{2}}(x_{kn}^\sigma)\mathbf{h}_s\left(\frac{1+x}{1+x_{kn}^\sigma}\right)\frac{u(x_{kn}^\sigma)}{1+x_{kn}^\sigma},$$

and set

$$\mathcal{H}_n := -\mathcal{H}_n^0 - \mathcal{H}_n^1 + 6\mathcal{H}_n^2 - 4\mathcal{H}_n^3.$$

The collocation-quadrature method seeks $u_n \in \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$ of the form

$$u_n(x) = \sqrt{\frac{1+x}{1-x}}p_n(x) = \upsilon^{-\frac{1}{2},\frac{1}{2}}(x)p_n(x), \quad p_n \in \mathbb{P}_n,$$

where u_n is the solution of the collocation system

$$\big(\mathcal{S}u_n + \mathcal{H}_n u_n\big)(x^\sigma_{jn}) = (\mathcal{J}f)(x^\sigma_{jn}), \quad j = 1, \cdots, n.$$

Let R_j be Chebyshev polynomials of third kind. Using

$$\mathcal{L}_n : \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_j \rangle_{\frac{1}{2},-\frac{1}{2}} \widetilde{p}_j, \quad \widetilde{p}_j = \upsilon^{-\frac{1}{2},\frac{1}{2}} R_j$$

and $\mathcal{M}^\sigma_n = \upsilon^{-\frac{1}{2},\frac{1}{2}}\mathcal{L}^\sigma_n \upsilon^{\frac{1}{2},-\frac{1}{2}}$, we can write the collocation system as

$$\mathcal{M}^\sigma_n(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n u_n = \mathcal{M}^\sigma_n \mathcal{J}f.$$

The collocation-quadrature method seeks $u_n \in \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$ of the form

$$u_n(x) = \sqrt{\frac{1+x}{1-x}} p_n(x) = \upsilon^{-\frac{1}{2},\frac{1}{2}}(x) p_n(x), \quad p_n \in \mathbb{P}_n,$$

where u_n is the solution of the collocation system

$$\big(\mathcal{S}u_n + \mathcal{H}_n u_n\big)(x^\sigma_{jn}) = (\mathcal{J}f)(x^\sigma_{jn}), \quad j = 1, \cdots, n.$$

Let R_j be Chebyshev polynomials of third kind. Using

$$\mathcal{L}_n : \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}, \quad u \mapsto \sum_{j=0}^{n-1} \langle u, \widetilde{p}_j \rangle_{\frac{1}{2},-\frac{1}{2}} \widetilde{p}_j, \quad \widetilde{p}_j = \upsilon^{-\frac{1}{2},\frac{1}{2}} R_j$$

and $\mathcal{M}^\sigma_n = \upsilon^{-\frac{1}{2},\frac{1}{2}} \mathcal{L}^\sigma_n \upsilon^{\frac{1}{2},-\frac{1}{2}}$, we can write the collocation system as

$$\mathcal{M}^\sigma_n(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n u_n = \mathcal{M}^\sigma_n \mathcal{J} f.$$

Definition

Let $\mathcal{A}_n : \mathrm{im}\ \mathcal{L}_n \longrightarrow \mathrm{im}\ \mathcal{L}_n$. We call the sequence $(\mathcal{A}_n)$ stable, if the operators are invertible for all sufficiently large $n \in \mathbb{N}$ and if the norms $\left\|(\mathcal{A}_n)^{-1}\mathcal{L}_n\right\|$ are uniformly bounded.

Lemma

We have the following strong convergences

$$\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n \longrightarrow \mathcal{S} + \mathcal{H} \quad \text{and} \quad \mathcal{M}_n^\sigma \mathcal{J} f \longrightarrow \mathcal{J} f$$

in the space $\mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$.

Proposition

If the collocation-quadrature method $(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ is stable, then there holds $u_n \longrightarrow u$ in $\mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$, where u_n is the solution of the collocation system and u is the solution of $(\mathcal{S} + \mathcal{H})u = \mathcal{J} f$.

Definition

Let $\mathcal{A}_n : \mathrm{im}\ \mathcal{L}_n \longrightarrow \mathrm{im}\ \mathcal{L}_n$. We call the sequence $(\mathcal{A}_n)$ stable, if the operators are invertible for all sufficiently large $n \in \mathbb{N}$ and if the norms $\left\|(\mathcal{A}_n)^{-1}\mathcal{L}_n\right\|$ are uniformly bounded.

Lemma

We have the following strong convergences

$$\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n \longrightarrow \mathcal{S}+\mathcal{H} \quad \textit{and} \quad \mathcal{M}_n^\sigma \mathcal{J} f \longrightarrow \mathcal{J} f$$

in the space $\mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$.

Proposition

If the collocation-quadrature method $(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n)$ *is stable, then there holds* $u_n \longrightarrow u$ *in* $\mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$, *where* u_n *is the solution of the collocation system and* u *is the solution of* $(\mathcal{S}+\mathcal{H})u = \mathcal{J} f$.

Definition

Let $\mathcal{A}_n : \mathrm{im}\ \mathcal{L}_n \longrightarrow \mathrm{im}\ \mathcal{L}_n$. We call the sequence $(\mathcal{A}_n)$ stable, if the operators are invertible for all sufficiently large $n \in \mathbb{N}$ and if the norms $\left\|(\mathcal{A}_n)^{-1}\mathcal{L}_n\right\|$ are uniformly bounded.

Lemma

We have the following strong convergences

$$\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n \longrightarrow \mathcal{S} + \mathcal{H} \quad \textit{and} \quad \mathcal{M}_n^\sigma \mathcal{J} f \longrightarrow \mathcal{J} f$$

in the space $\mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$.

Proposition

If the collocation-quadrature method $(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ *is stable, then there holds* $u_n \longrightarrow u$ *in* $\mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}}$*, where* u_n *is the solution of the collocation system and* u *is the solution of* $(\mathcal{S} + \mathcal{H})u = \mathcal{J} f$*.*

The following steps show how to investigate the stability of the collocation-quadrature method.

1. Construction of a suitable C^*-algebra $\mathfrak{F}$ and a ideal $\mathfrak{J} \subset \mathfrak{F}$, where

$$\mathfrak{F} := \Big\{ (\mathcal{A}_n) : \mathcal{A}_n : \operatorname{im} \mathcal{L}_n \longrightarrow \operatorname{im} \mathcal{L}_n, \\ \exists\, \mathcal{W}_t(\mathcal{A}_n) : \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}},\ t = 1, 2, \\ \exists\, \mathcal{W}_t(\mathcal{A}_n) : \ell^2(\mathbb{N}_0) \longrightarrow \ell^2(\mathbb{N}_0),\ t = 3, 4 \Big\}.$$

2. Application of SILBERMANN'S lifting Theorem gives

$$(\mathcal{A}_n) \in \mathfrak{F} \text{ is stable } \iff \mathcal{W}_t(\mathcal{A}_n),\ t \in \{1, 2, 3, 4\} \text{ and } \\ (\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J} \text{ are invertible.}$$

3. Proving that $(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ and $(\mathcal{M}_n^\sigma a \mathcal{L}_n)$, $a \in \mathbf{C}[-1, 1]$ belong to $\mathfrak{F}$.

The following steps show how to investigate the stability of the collocation-quadrature method.

1. Construction of a suitable C^*-algebra $\mathfrak{F}$ and a ideal $\mathfrak{J} \subset \mathfrak{F}$, where

$$\mathfrak{F} := \Big\{ (\mathcal{A}_n) : \mathcal{A}_n : \mathrm{im}\, \mathcal{L}_n \longrightarrow \mathrm{im}\, \mathcal{L}_n, \\ \exists\, \mathcal{W}_t(\mathcal{A}_n) : \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}},\ t = 1, 2, \\ \exists\, \mathcal{W}_t(\mathcal{A}_n) : \ell^2(\mathbb{N}_0) \longrightarrow \ell^2(\mathbb{N}_0),\ t = 3, 4 \Big\}.$$

2. Application of SILBERMANN'S lifting Theorem gives

$$(\mathcal{A}_n) \in \mathfrak{F} \text{ is stable } \iff \mathcal{W}_t(\mathcal{A}_n),\ t \in \{1, 2, 3, 4\} \text{ and } \\ (\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J} \text{ are invertible.}$$

3. Proving that $(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ and $(\mathcal{M}_n^\sigma a \mathcal{L}_n)$, $a \in \mathbf{C}[-1, 1]$ belong to $\mathfrak{F}$.

The following steps show how to investigate the stability of the collocation-quadrature method.

1. Construction of a suitable C^*-algebra $\mathfrak{F}$ and a ideal $\mathfrak{J} \subset \mathfrak{F}$, where

$$\mathfrak{F} := \Big\{(\mathcal{A}_n) : \mathcal{A}_n : \mathrm{im}\ \mathcal{L}_n \longrightarrow \mathrm{im}\ \mathcal{L}_n,$$
$$\exists\, \mathcal{W}_t(\mathcal{A}_n) : \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}} \longrightarrow \mathbf{L}^2_{\frac{1}{2},-\frac{1}{2}},\ t = 1, 2,$$
$$\exists\, \mathcal{W}_t(\mathcal{A}_n) : \ell^2(\mathbb{N}_0) \longrightarrow \ell^2(\mathbb{N}_0),\ t = 3, 4\Big\}.$$

2. Application of Silbermann's lifting Theorem gives

$$(\mathcal{A}_n) \in \mathfrak{F} \text{ is stable } \iff \mathcal{W}_t(\mathcal{A}_n),\ t \in \{1, 2, 3, 4\} \text{ and}$$
$$(\mathcal{A}_n) + \mathfrak{J} \in \mathfrak{F}/\mathfrak{J} \text{ are invertible.}$$

3. Proving that $(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ and $(\mathcal{M}_n^\sigma a \mathcal{L}_n)$, $a \in \mathbf{C}[-1, 1]$ belong to $\mathfrak{F}$.

Let $\mathfrak{A}$ be the smallest closed C^*-subalgebra of $\mathfrak{F}$, which contains $\mathfrak{J}$,

$$(\mathcal{M}_n^\sigma a \mathcal{L}_n) \quad \text{and} \quad (\mathcal{M}_n^\sigma \mathcal{S} \mathcal{L}_n) \quad \text{as well as} \quad (\mathcal{M}_n^\sigma \mathcal{H}_n^s \mathcal{L}_n)$$

for all $s \in \{0, 1, 2, 3\}$.

4. Using the locale principle of ALLAN UND DOUGLAS to prove

$$(\mathcal{A}_n) \in \mathfrak{A} \text{ is stable} \iff \mathcal{W}_t(\mathcal{A}_n),\ t \in \{1, 2, 3, 4\} \text{ invertible.}$$

5. Checking the invertibility of $\mathcal{W}_t(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ for $t \in \{1, 2, 3, 4\}$.

Let $\mathfrak{A}$ be the smallest closed C^*-subalgebra of $\mathfrak{F}$, which contains $\mathfrak{J}$,

$$(\mathcal{M}_n^\sigma a \mathcal{L}_n) \quad \text{and} \quad (\mathcal{M}_n^\sigma \mathcal{S} \mathcal{L}_n) \quad \text{as well as} \quad (\mathcal{M}_n^\sigma \mathcal{H}_n^s \mathcal{L}_n)$$

for all $s \in \{0, 1, 2, 3\}$.

4. Using the locale principle of Allan und Douglas to prove

$$(\mathcal{A}_n) \in \mathfrak{A} \text{ is stable} \iff \mathcal{W}_t(\mathcal{A}_n),\ t \in \{1, 2, 3, 4\} \text{ invertible.}$$

5. Checking the invertibility of $\mathcal{W}_t(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ for $t \in \{1, 2, 3, 4\}$.

Let $\mathfrak{A}$ be the smallest closed C^*-subalgebra of $\mathfrak{F}$, which contains $\mathfrak{J}$,

$$(\mathcal{M}_n^\sigma a \mathcal{L}_n) \quad \text{and} \quad (\mathcal{M}_n^\sigma \mathcal{S} \mathcal{L}_n) \quad \text{as well as} \quad (\mathcal{M}_n^\sigma \mathcal{H}_n^s \mathcal{L}_n)$$

for all $s \in \{0, 1, 2, 3\}$.

4. Using the locale principle of Allan und Douglas to prove

$$(\mathcal{A}_n) \in \mathfrak{A} \text{ is stable} \iff \mathcal{W}_t(\mathcal{A}_n),\ t \in \{1, 2, 3, 4\} \text{ invertible.}$$

5. Checking the invertibility of $\mathcal{W}_t(\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n)$ for $t \in \{1, 2, 3, 4\}$.

Proposition

$(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n)$ is stable if and only if

(i) $\mathcal{W}_1(\mathcal{A}_n) = \mathcal{S}+\mathcal{H}$,

(ii) $\mathcal{W}_2(\mathcal{A}_n) = \mathbf{i}\mathcal{I}$,

(iii) $\mathcal{W}_3(\mathcal{A}_n) = \mathbf{T}+\mathbf{H}$,

(iv) $\mathcal{W}_4(\mathcal{A}_n) = -\mathbf{T}+\mathbf{H}+\mathbf{B}$

are invertible, where $\mathcal{I}$ is the Identity and where

$$\mathbf{T} = \left[\frac{1-(-1)^{j-k}}{\pi\mathbf{i}(j-k)}\right]_{j,k=0}^{\infty}, \quad \mathbf{H} = \left[\frac{1-(-1)^{j+k+1}}{\pi\mathbf{i}(j+k+1)}\right]_{j,k=0}^{\infty}$$

as well as

$$\mathbf{B} = \left[2\,\mathbf{h}\left(\left[\frac{j+1}{k+1}\right]^2\right)\frac{1}{k+1}\right]_{j,k=0}^{\infty}.$$

Theorem

The collocation-quadrature method $(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n)$ is stable if and only if the operator $-\mathbf{T}+\mathbf{H}+\mathbf{B}$ has a trivial null space.

Proposition

$(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n)$ *is stable if and only if*

(i) $\mathcal{W}_1(\mathcal{A}_n) = \mathcal{S}+\mathcal{H}$,

(ii) $\mathcal{W}_2(\mathcal{A}_n) = \mathbf{i}\mathcal{I}$,

(iii) $\mathcal{W}_3(\mathcal{A}_n) = \mathbf{T}+\mathbf{H}$,

(iv) $\mathcal{W}_4(\mathcal{A}_n) = -\mathbf{T}+\mathbf{H}+\mathbf{B}$

are invertible, where $\mathcal{I}$ *is the Identity and where*

$$\mathbf{T} = \left[\frac{1-(-1)^{j-k}}{\pi\mathbf{i}(j-k)}\right]_{j,k=0}^{\infty}, \quad \mathbf{H} = \left[\frac{1-(-1)^{j+k+1}}{\pi\mathbf{i}(j+k+1)}\right]_{j,k=0}^{\infty}$$

as well as

$$\mathbf{B} = \left[2\,\mathbf{h}\left(\left[\frac{j+1}{k+1}\right]^2\right)\frac{1}{k+1}\right]_{j,k=0}^{\infty}.$$

Theorem

The collocation-quadrature method $(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n)$ *is stable if and only if the operator* $-\mathbf{T}+\mathbf{H}+\mathbf{B}$ *has a trivial null space.*

For $f = -\mathbf{i}$ the collocation system is given by

$$\mathcal{M}_n^\sigma(\mathcal{S} + \mathcal{H}_n)\mathcal{L}_n u_n = \mathcal{M}_n^\sigma \mathcal{J} f = \mathcal{M}_n^\sigma(-\mathbf{i}\sqrt{1+\cdot}). \tag{3}$$

The respective system of linear equations is solved with the Krylov subspace method CGNR. The system matrix and the right-hand side are given by

$$\mathbb{A}_n := \left[\sqrt{\frac{1 - x_{jn}^\sigma}{1 - x_{kn}^\sigma}} \left(\mathcal{S}\widetilde{\ell}_{kn}^\sigma + \mathcal{H}_n \widetilde{\ell}_{kn}^\sigma \right) (x_{jn}^\sigma) \right]_{j,k=1}^n$$

and

$$\left[-\mathbf{i}\sqrt{1 - x_{jn}^\tau} \right]_{j=1}^n,$$

respectively, where ℓ_{kn}^σ denote the usual Lagrange base polynomials and

$$\widetilde{\ell}_{kn}^\sigma(x) = v^{-\frac{1}{2},\frac{1}{2}}(x)\ell_{kn}^\sigma(x)v^{\frac{1}{2},-\frac{1}{2}}(x_{kn}^\sigma).$$

Lemma

The sequence $\left(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n\right)$ *is stable if and only if the matrices* $\mathbb{A}_n$ *are invertible for all sufficiently large* $n \in \mathbb{N}$ *and if their condition numbers are uniformly bounded.*

We have the following computational results.

n	$\mathrm{cond}(\mathbb{A}_n)$	s_1	s_2	s_3	s_n
128	3.1668	0.3198	0.4961	0.6698	1.0127
256	3.3061	0.3063	0.4596	0.6199	1.0127
512	3.4221	0.2959	0.4303	0.5776	1.0127
1024	3.5194	0.2877	0.4064	0.5416	1.0127
2048	3.6016	0.2812	0.3867	0.5109	1.0127
4096	3.6716	0.2758	0.3702	0.4845	1.0127
8192	3.7315	0.2714	0.3563	0.4616	1.0127
16384	3.7831	0.2677	0.3444	0.4417	1.0127

Table 1: Collocation-quadrature (3)

s_1, s_2, s_3 denote the three smallest singular values and s_n the greatest singular value of $\mathbb{A}_n$.

Lemma

The sequence $\left(\mathcal{M}_n^\sigma(\mathcal{S}+\mathcal{H}_n)\mathcal{L}_n\right)$ *is stable if and only if the matrices* $\mathbb{A}_n$ *are invertible for all sufficiently large* $n \in \mathbb{N}$ *and if their condition numbers are uniformly bounded.*

We have the following computational results.

n	$\mathrm{cond}(\mathbb{A}_n)$	s_1	s_2	s_3	s_n
128	3.1668	0.3198	0.4961	0.6698	1.0127
256	3.3061	0.3063	0.4596	0.6199	1.0127
512	3.4221	0.2959	0.4303	0.5776	1.0127
1024	3.5194	0.2877	0.4064	0.5416	1.0127
2048	3.6016	0.2812	0.3867	0.5109	1.0127
4096	3.6716	0.2758	0.3702	0.4845	1.0127
8192	3.7315	0.2714	0.3563	0.4616	1.0127
16384	3.7831	0.2677	0.3444	0.4417	1.0127

Table 1: Collocation-quadrature (3)

s_1, s_2, s_3 denote the three smallest singular values and s_n the greatest singular value of $\mathbb{A}_n$.

Thanks for your attention!