Linearizations of polynomial and rational matrices

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Kalamata, July 2-6, 2018

Polynomial and rational eigenvalue problems

- Polynomial and rational $m \times n$ matrices with matrix coefficients $P_i, i = 0, \ldots, d$ and A, B, C

$$
\begin{aligned}
P(\lambda) &= \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0 \\
R(\lambda) &= P(\lambda) + C(\lambda I_\ell - A)^{-1} B
\end{aligned}
$$

- Want to compute the complete eigenstructure
  - Finite elementary divisors
  - Infinite elementary divisors
  - Left and right null space structure
- Many application areas:
  - Vibrating systems
  - Electrical circuits
  - Dynamical systems

What I'll talk about

- Companion and Fiedler matrices (regular)
- Block Kronecker linearizations (general)
- Dual minimal bases
- Structure preserving polynomial backward stability
- Structure preserving rational backward stability
- Extensions

Companion and Fiedler matrices (scalar polynomials)

- The roots of the monic polynomial

$$p(\lambda) = \lambda^d + \lambda^{d-1} p_{d-1} + \ldots + \lambda p_1 + p_0$$

are the eigenvalues of the companion matrix (Frobenius)

$$\lambda I_d - C; \quad C =: \begin{bmatrix} -p_{d-1} & \ldots & -p_1 & -p_0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

- Fiedler extended this to a new family of matrices with the same elements, but rearranged in a strange way

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Companion matrix proof

The structure of

$$\lambda I_d - C = \begin{bmatrix} \lambda + p_{d-1} & \dots & p_1 & p_0 \\ -1 & \lambda & & \\ & \ddots & \ddots & \\ & & -1 & \lambda \end{bmatrix}$$

yields

$$(\lambda I_d - C) \begin{bmatrix} \lambda^{d-1} \\ \vdots \\ \lambda \\ 1 \end{bmatrix} = \begin{bmatrix} p(\lambda) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

from which it follows that $p(\lambda) = \det(\lambda I_d - C)$

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Companion and Fiedler matrices

- The so-called Fiedler matrices can be constructed from products of elementary factors of the type

$$A_k = \begin{bmatrix} I_{d-k-1} & & \\ & C_k & \\ & & I_{k-1} \end{bmatrix}, \quad C_k = \begin{bmatrix} -p_k & -1 \\ 1 & 0 \end{bmatrix}.$$

- and typically contain a staircase of elements p_k, e.g. $(d = 4)$:

$$\lambda I_4 - F := \begin{bmatrix} \lambda + p_3 & -1 & 0 & 0 \\ p_2 & \lambda & p_1 & p_0 \\ -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}.$$

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Permuted Fiedler matrices

- If we permute the staircase to the top left corner, and scale we can obtain the following block anti-triangular form

$$\lambda B + A := \left[\begin{array}{ccc|c} \lambda + p_3 & & 0 & 1 \\ p_2 & p_1 & p_0 & -\lambda \\ \hline 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \end{array}\right],$$

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Block Kronecker pencils

- We start from the definitions

$$K_k(\lambda) = \left.\begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \end{bmatrix}\right\} k, \quad \Pi_k(\lambda) = \left.\begin{bmatrix} \lambda^k \\ \vdots \\ \lambda \\ 1 \end{bmatrix}\right\} k+1$$

  and the equation $K_k(\lambda)\Pi_k(\lambda) = 0$, implying that the rows of $K_k(\lambda)$ are dual to the columns of $\Pi_k(\lambda)$.

- A general block Kronecker pencil with $d = \epsilon + \eta + 1$ is then is of the form

$$L(\lambda) := \lambda B + A = \begin{array}{c} \left[\begin{array}{c|c} \lambda M_1 + M_0 & K_\eta^T(\lambda) \otimes I_m \\ \hline K_\epsilon(\lambda) \otimes I_n & 0 \end{array}\right] \begin{array}{l} \}\,(\eta+1)m \\ \}\,\epsilon n \end{array} \\ \underbrace{\qquad\qquad}_{(\epsilon+1)n} \quad \underbrace{\qquad\qquad}_{\eta m} \end{array}$$

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A simple transformation (scalar case)

- $K_k(\lambda)$ can be embedded in a unimodular matrix

$$U_k(\lambda) := \left.\begin{bmatrix} 1 & -\lambda & & \\ & \ddots & \ddots & \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix}\right\} k+1$$

- with unimodular inverse $V_k(\lambda)$ whose last column is $\Pi_k(\lambda)$

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A simple transformation

Now apply this to $\lambda B + A$ (scalar case for simplicity):

$$\left[\begin{array}{c|c} V_\eta^T(\lambda)\otimes I_m & \\ \hline 0 & I_{\epsilon n}\end{array}\right]\left[\begin{array}{c|c} \lambda M_1 + M_0 & K_\eta^T(\lambda)\otimes I_m \\ \hline K_\epsilon(\lambda)\otimes I_n & 0\end{array}\right]\left[\begin{array}{c|c} V_\epsilon(\lambda)\otimes I_n & \\ \hline 0 & I_{\eta m}\end{array}\right]$$

$$= \left[\begin{array}{cc|c} X(\lambda) & Y(\lambda) & I_{\eta m} \\ Z(\lambda) & P(\lambda) & 0 \\ \hline I_{\epsilon n} & 0 & 0\end{array}\right]$$

where $P(\lambda) = (\Pi_\eta^T(\lambda)\otimes I_m)(\lambda M_1 + M_0)(\Pi_\epsilon(\lambda)\otimes I_n)$ provided

$$\lambda M_1 + M_0 = \begin{bmatrix} \lambda P_d & \iddots & \iddots \\ \iddots & \iddots & \iddots \\ \iddots & \iddots & P_0 \end{bmatrix},$$

$$\sum_{i+j=d-k+2} [M_1]_{i,j} + \sum_{i+j=d-k+1} [M_0]_{i,j} = P_k,\ k \in 0:d,\ i \in 1:\epsilon+1,\ j \in 1:\eta+1$$

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Example (matrix case)

Two possible linearizations for an $m \times n$ quartic polynomial matrix

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0$$

$$\lambda B + A = \left[\begin{array}{ccc|c} \lambda P_4 & \lambda P_3 + P_2 & 0 & I_m \\ 0 & P_1 & P_0 & -\lambda I_m \\ \hline I_n & -\lambda I_n & 0 & 0 \\ 0 & I_n & -\lambda I_n & 0 \end{array}\right],$$

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Some properties

- Block Kronecker pencil linearizations "contain" companion and Fiedler matrices and Fiedler like pencils as special cases
- Their finite and infinite elementary divisors are those of $P(\lambda)$ because the linearizations are "strong" (the reversed pencil linearizes the reversed polynomial matrix)

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Polynomial backward stability

- We must assume that the matrices were scaled such that

$$\|P(\cdot)\|_F^2 := \sum_i \|P_i\|_F^2 = \mathcal{O}(1) \implies \|(A,B)\|_F^2 := \|A\|_F^2 + \|B\|_F^2 = \mathcal{O}(1)$$

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- and there exists an *equivalent* $\|(\Delta M_0, \Delta M_1)\|_F = \mathcal{O}(\varepsilon)$ such that $\widehat{M}_0 := M_0 + \Delta M_0$, $\widehat{M}_1 := M_1 + \Delta M_1$ and

$$(I+X)\left[L(\lambda)+\Delta L(\lambda)\right](I+Y) = \left[\begin{array}{c|c} \lambda \widehat{M}_1 + \widehat{M}_0 & K_\eta^T(\lambda)\otimes I_m \\ \hline K_\epsilon(\lambda)\otimes I_n & 0 \end{array}\right]$$

with $\|(X,Y)\|_F = \mathcal{O}(\varepsilon)$, which implies that this approach is *structurally* backward stable if $\|P(\cdot)\|_F = \mathcal{O}(1)$

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Proof of backward stability to a nearby polynomial matrix $P(\lambda) + \Delta P(\lambda)$ provided we scaled the coefficient matrix to 1 :

First we restore the anti-triangular structure by strict equivalence

$$\left[\begin{array}{cc} I_{(\eta+1)m} & 0 \\ C & I_{\epsilon n} \end{array}\right] \left(\left[\begin{array}{c|c} \lambda M_1 + M_0 & K_\eta^T(\lambda) \otimes I_m \\ \hline K_\epsilon(\lambda) \otimes I_n & 0 \end{array}\right] + \left[\begin{array}{c|c} \lambda \Delta B_{11} + \Delta A_{11} & \lambda \Delta B_{12} + \Delta A_{12} \\ \hline \lambda \Delta B_{21} + \Delta A_{21} & \lambda \Delta B_{22} + \Delta A_{22} \end{array}\right] \right) \left[\begin{array}{cc} I_{(\epsilon+1)n} & D \\ 0 & I_{\eta m} \end{array}\right] =$$

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This is a nonlinear system of equations $(C, D) = f(A, B, \Delta A, \Delta B)$, but has a solution of norm $\|(C, D)\|_F = \mathcal{O}(\varepsilon)$.

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Polynomial backward stability

Second, we "restore" the perturbed dual minimal bases

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Bounding $\|(C, D)\|_F$, $\|(X_\epsilon, Y_\epsilon)\|_F$ and $\|(X_\eta, Y_\eta)\|_F$, we can prove

Theorem

Let $A + \lambda B$ be a block Kronecker linearization of a polynomial matrix $P(\lambda)$ and let $\Delta A + \lambda \Delta B$ be the backward error induced by the eigenstructure algorithm. Then the corresponding backward error $\Delta P(\cdot)$ has a norm satisfying

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The rational case

It follows from realization theory that the "zero structure" of a rational matrix

$$R(\lambda) := P(\lambda) + C(\lambda I_\ell - A)^{-1}B,$$

where

$$P(\lambda) := \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0$$

is the same as the "zero structure" of the so-called *system matrix*

$$S(\lambda) := \begin{bmatrix} P(\lambda) & C \\ B & A - \lambda I_\ell \end{bmatrix}$$

provided (A, B) is *controllable* and (A, C) is *observable*.

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Now linearize $S(\lambda)$ [ADMZ2018]

The rational matrix

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has the same "zero structure" as

$$L(\lambda) := \begin{bmatrix} M(\lambda) & N_\eta^T C & K_\eta^T(\lambda) \\ BN_\epsilon & A - \lambda I_\ell & 0 \\ K_\epsilon(\lambda) & 0 & 0 \end{bmatrix}.$$

where

$$\left[\begin{array}{c} K_k(\lambda) \\ \hline N_k \end{array}\right] = \left[\begin{array}{cccc} I & -\lambda I & & \\ & \ddots & \ddots & \\ & & I & -\lambda I \\ \hline 0 & \dots & 0 & I \end{array}\right]$$

Proof [ADMZ2018]

There exist unimodular transformations $V_\epsilon(\lambda)$ and $V_\eta(\lambda)$ such that

$$\begin{bmatrix} V_\eta^T(\lambda) & 0 \\ 0 & I_{\epsilon n} \end{bmatrix} L(\lambda) \begin{bmatrix} V_\epsilon(\lambda) & 0 \\ 0 & I_{\eta m} \end{bmatrix} = \left[\begin{array}{cc|cc} X(\lambda) & Y(\lambda) & 0 & I_{\eta m} \\ Z(\lambda) & P(\lambda) & C & 0 \\ \hline 0 & B & A - \lambda I_\ell & 0 \\ I_{\epsilon n} & 0 & 0 & 0 \end{array}\right]$$

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Example

No operations are needed : the $m \times n$ rational matrix

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Applying a stable algorithm to

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will introduce an error $\Delta L(\lambda)$ that destroys its structure
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Scaling of $R(\lambda)$ [DQV2018]

There exists a scaling $R_s(\lambda_s) := d_r R(d_\lambda \lambda)$ which guarantees that

$$\|D_{ata}\|_F = \mathcal{O}(1), \quad \text{where} \quad D_{ata} = (A, B, C, P_0, \ldots, P_d)$$

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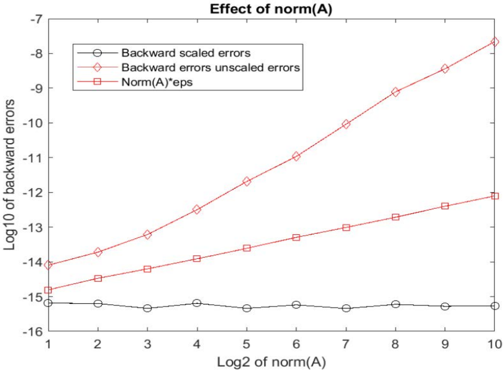

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Block Kronecker ℓ-ifications

For $\ell = 2$ we can obtain "quadratifications" of an even polynomial matrix as follows

$$P(\lambda) = \lambda^6 P_6 + \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 \,.$$

$$\lambda^2 C + \lambda B + A = \left[\begin{array}{cc|c} \lambda^2 P_6 + \lambda P_5 + P_4 & \lambda P_3/2 & I_m \\ \lambda P_3/2 & \lambda^2 P_2 + \lambda P_1 + P_0 & -\lambda^2 I_m \\ \hline I_n & -\lambda^2 I_n & 0 \end{array}\right].$$

This is also a symmetric "quadratification" if $P(\lambda)$ is symmetric

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References

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