Linearizations of polynomial and rational matrices

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Kalamata, July 2-6, 2018
Polynomial and rational eigenvalue problems

- Polynomial and rational $m \times n$ matrices with matrix coefficients $P_i, i = 0, \ldots, d$ and $A, B, C$

\[
P(\lambda) = \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0
\]

\[
R(\lambda) = P(\lambda) + C(\lambda I - A)^{-1} B
\]

- Want to compute the complete eigenstructure
  - Finite elementary divisors
  - Infinite elementary divisors
  - Left and right null space structure

- Many application areas:
  - Vibrating systems
  - Electrical circuits
  - Dynamical systems
What I’ll talk about

- Companion and Fiedler matrices (regular)
- Block Kronecker linearizations (general)
- Dual minimal bases
- Structure preserving polynomial backward stability
- Structure preserving rational backward stability
- Extensions
Companion and Fiedler matrices (scalar polynomials)

- The roots of the monic polynomial

\[ p(\lambda) = \lambda^d + \lambda^{d-1} p_{d-1} + \ldots + \lambda p_1 + p_0 \]

are the eigenvalues of the companion matrix (Frobenius)

\[ \lambda I_d - C; \quad C = \begin{bmatrix}
- p_{d-1} & \ldots & - p_1 & - p_0 \\
1 & 0 & \cdot & \cdot & \cdot \\
& & \ldots & \cdot & \cdot \\
& & & 1 & 0
\end{bmatrix} \]

- Fiedler extended this to a new family of matrices with the same elements, but rearranged in a strange way
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\[ \lambda I_d - C; \quad C = \begin{bmatrix} -p_{d-1} & \ldots & -p_1 & -p_0 \\ 1 & 0 & \ddots & \ddots \\ & \ddots & \ddots & 1 \\ & & 1 & 0 \end{bmatrix} \]

- Fiedler extended this to a new family of matrices with the same elements, but rearranged in a strange way
Companion matrix proof

The structure of

\[
\lambda I_d - C = \begin{bmatrix}
\lambda + p_{d-1} & \cdots & p_1 & p_0 \\
-1 & \lambda \\
& \ddots & \ddots \\
& & -1 & \lambda
\end{bmatrix}
\]

yields

\[
(\lambda I_d - C) \begin{bmatrix}
\lambda^{d-1} \\
\vdots \\
\lambda \\
1
\end{bmatrix} = \begin{bmatrix}
p(\lambda) \\
0 \\
\vdots \\
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\end{bmatrix}
\]

from which it follows that \( p(\lambda) = \det(\lambda I_d - C) \)
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Companion and Fiedler matrices

The so-called Fiedler matrices can be constructed from products of elementary factors of the type

\[ A_k = \begin{bmatrix} I_{d-k-1} & C_k \\ l_{k-1} & C_k \end{bmatrix}, \quad C_k = \begin{bmatrix} -p_k & -1 \\ 1 & 0 \end{bmatrix}. \]

and typically contain a staircase of elements \( p_k \), e.g. \((d = 4)\):

\[ \lambda l_4 - F := \begin{bmatrix} \lambda + p_3 & -1 & 0 & 0 \\ p_2 & \lambda & p_1 & p_0 \\ -1 & 0 & \lambda & 0 \\ 0 & 0 & -1 & \lambda \end{bmatrix}. \]
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but the derivation is tedious (show that \( F \) and \( C \) are similar).
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- but the derivation is tedious (show that \( F \) and \( C \) are similar)
Permuted Fiedler matrices

- If we permute the staircase to the top left corner, and scale we can obtain the following block anti-triangular form

\[
\lambda B + A \coloneqq \begin{bmatrix}
\lambda + p_3 & 0 & 1 \\
p_2 & p_1 & p_0 & -\lambda \\
1 & -\lambda & 0 & 0 \\
0 & 1 & -\lambda & 0
\end{bmatrix},
\]

- where the anti-diagonal blocks look like Kronecker blocks
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- We will show that the identity

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\det(\lambda B + A) = p(\lambda)
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easily follows from this permuted form
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- Moreover, it extends to general non-monic matrix polynomials
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easily follows from this permuted form
- Moreover, it extends to general non-monic matrix polynomials
Block Kronecker pencils

We start from the definitions

$$K_k(\lambda) = \begin{bmatrix} 1 & -\lambda \\ \vdots & \ddots & \ddots \\ & & 1 & -\lambda \end{bmatrix} \text{ for } k, \quad \Pi_k(\lambda) = \begin{bmatrix} \lambda^k \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \text{ for } k+1$$

and the equation $K_k(\lambda)\Pi_k(\lambda) = 0$, implying that the rows of $K_k(\lambda)$ are dual to the columns of $\Pi_k(\lambda)$.

A general block Kronecker pencil with $d = \epsilon + \eta + 1$ is then of the form

$$L(\lambda) := \lambda B + A = \begin{bmatrix} \lambda M_1 + M_0 & K^T_\eta(\lambda) \otimes I_m \\ K_\epsilon(\lambda) \otimes I_n & 0 \end{bmatrix} \left\{ \begin{array}{c} (\eta+1)m \\ \epsilon n \end{array} \right\}_{(\epsilon+1)n \ \eta m}$$
Block Kronecker pencils

- We start from the definitions

\[ K_k(\lambda) = \begin{bmatrix} 1 & -\lambda \\ & \ddots & \ddots \\ & & 1 & -\lambda \end{bmatrix} \quad \text{and} \quad \Pi_k(\lambda) = \begin{bmatrix} \lambda^k \\ & \vdots \\ & & \lambda \\ & & 1 \end{bmatrix} \]

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- A general block Kronecker pencil with \( d = \epsilon + \eta + 1 \) is then of the form

\[
L(\lambda) := \lambda B + A = \begin{bmatrix} \lambda M_1 + M_0 & K_\eta^T(\lambda) \otimes I_m \\ \frac{K_\epsilon(\lambda) \otimes I_n}{(\epsilon+1)n} & 0 \end{bmatrix} \begin{bmatrix} \lambda^{\eta+1}m \\ \epsilon n \end{bmatrix}
\]

\( \sqrt{\epsilon+1}n \) \quad \text{and} \quad \eta m
A simple transformation (scalar case)

- $K_k(\lambda)$ can be embedded in a unimodular matrix

$$U_k(\lambda) := \begin{bmatrix}
1 & -\lambda \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & 1 & -\lambda \\
& & & 1
\end{bmatrix}^{k+1}$$

- with unimodular inverse $V_k(\lambda)$ whose last column is $\Pi_k(\lambda)$

$$U_k^{-1}(\lambda)e_{k+1} = V_k(\lambda)e_{k+1} = \Pi_k(\lambda)$$
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- and compresses the columns of $K_k(\lambda)$ to a simple form

$$K_k(\lambda)V_k(\lambda) = \begin{bmatrix} I_k & 0 \end{bmatrix}.$$
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$$U_k(\lambda) := \begin{bmatrix} 1 & -\lambda \\ \vdots & \vdots & \ddots & \ddots \\ & & & & 1 & -\lambda \\ & & & & & 1 \\ & & & & & & 1 \end{bmatrix}_{k+1}$$

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- and compresses the columns of $K_k(\lambda)$ to a simple form

$$K_k(\lambda)V_k(\lambda) = \begin{bmatrix} I_k \\ 0 \end{bmatrix}.$$
A simple transformation

Now apply this to $\lambda B + A$ (scalar case for simplicity):

$$\begin{bmatrix}
V_\eta^T(\lambda) \otimes I_m \\
\lambda M_1 + M_0 \\
K_\epsilon(\lambda) \otimes I_n \\
V_\epsilon(\lambda) \otimes I_n
\end{bmatrix}
\begin{bmatrix}
\lambda M_1 + M_0 \\
K_\eta^T(\lambda) \otimes I_m \\
K_\epsilon(\lambda) \otimes I_n \\
0
\end{bmatrix}
\begin{bmatrix}
V_\epsilon(\lambda) \otimes I_n \\
0 \\
I_{\eta m}
\end{bmatrix}$$

$$= \begin{bmatrix}
X(\lambda) & Y(\lambda) & I_{\eta m} \\
Z(\lambda) & P(\lambda) & 0 \\
I_{\epsilon n} & 0 & 0
\end{bmatrix}$$

where $P(\lambda) = (\Pi^T_\eta(\lambda) \otimes I_m)(\lambda M_1 + M_0)(\Pi_\epsilon(\lambda) \otimes I_n)$ provided

$$\lambda M_1 + M_0 = \begin{bmatrix}
\lambda P_d & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & P_0
\end{bmatrix},$$

$$\sum_{i+j=d-k+2} [M_1]_{i,j} + \sum_{i+j=d-k+1} [M_0]_{i,j} = P_k, \ k \in 0 : d, \ i \in 1 : \epsilon+1, \ j \in 1 : \eta+1$$
A simple transformation

Now apply this to $\lambda B + A$ (scalar case for simplicity):

\[
\begin{bmatrix}
V_{\eta}^T(\lambda) \otimes I_m & \lambda M_1 + M_0 & K_{\eta}^T(\lambda) \otimes I_m \\
0 & K_{\epsilon}(\lambda) \otimes I_n & 0
\end{bmatrix}
\begin{bmatrix}
V_{\epsilon}(\lambda) \otimes I_n \\
0 & I_{\eta m}
\end{bmatrix}
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\lambda M_1 + M_0 = \begin{bmatrix}
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\sum_{i+j=d-k+2} [M_1]_{i,j} + \sum_{i+j=d-k+1} [M_0]_{i,j} = P_k, \ k \in 0:d, \ i \in 1:\epsilon+1, \ j \in 1:\eta+1
\]
Since

$$\Pi_\eta(\lambda) \Pi_\epsilon^T(\lambda) = \begin{bmatrix} \lambda^{\epsilon+\eta} & \cdots & \cdots & \cdots & \lambda^\eta \\ \\ \vdots & & & & \\ \vdots & & & & \\ \vdots & \cdots & \cdots & \cdots & \lambda^2 \\ \lambda^\epsilon & \cdots & \lambda^2 & \lambda & 1 \end{bmatrix}$$

\(\lambda M_1 + M_0\) must have a particular block structure

$$\lambda M_1 + M_0 = \begin{bmatrix} \lambda P_d & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & P_0 \end{bmatrix},$$

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Since

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\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \lambda^2 \\
\vdots & \ddots & \ddots & \lambda^2 & \lambda \\
\lambda^{\epsilon} & \ldots & \lambda^2 & \lambda & 1 
\end{bmatrix}$$

\(\lambda M_1 + M_0\) must have a particular block structure

$$\lambda M_1 + M_0 = \begin{bmatrix} 
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Example (matrix case)

Two possible linearizations for an $m \times n$ quartic polynomial matrix

$$P(\lambda) = \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0$$

$$\lambda B + A = \begin{bmatrix}
\lambda P_4 & \lambda P_3 + P_2 & 0 & l_m \\
0 & P_1 & P_0 & -\lambda l_m \\
l_n & -\lambda l_n & 0 & 0 \\
0 & I_n & -\lambda l_n & 0 \\
\end{bmatrix},$$

$$\lambda B + A = \begin{bmatrix}
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I_n & -\lambda I_n & 0 & 0 \\
0 & I_n & -\lambda I_n & 0
\end{bmatrix}
\]
Some properties

- Block Kronecker pencil linearizations “contain” companion and Fiedler matrices and Fiedler like pencils as special cases
- Their finite and infinite elementary divisors are those of $P(\lambda)$ because the linearizations are “strong”
  (the reversed pencil linearizes the reversed polynomial matrix)
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- We can preserve symmetries of odd-degree $P(\lambda)$. 
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- Their finite and infinite elementary divisors are those of $P(\lambda)$ because the linearizations are “strong” (the reversed pencil linearizes the reversed polynomial matrix).
- We can preserve symmetries of odd-degree $P(\lambda)$.
- The BK-linearization of a singular $P(\lambda)$ is also singular and their left and right nullspace structures are shifted by $\epsilon$ and $\eta$. 
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- The backward error induced by the eigenstructure algorithms (staircase algorithm followed by the QZ algorithm) can be mapped back to the coefficients of $P(\lambda)$. 
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- The backward error induced by the eigenstructure algorithms (staircase algorithm followed by the QZ algorithm) can be mapped back to the coefficients of $P(\lambda)$.
Polynomial backward stability

- We must assume that the matrices were scaled such that
  \[ \| P(\cdot) \|_F^2 := \sum_i \| P_i \|_F^2 = O(1) \implies \| (A, B) \|_F^2 := \| A \|_F^2 + \| B \|_F^2 = O(1) \]

- The eigenstructure algorithms applied to a BK pencil
  \[ L(\lambda) := \lambda B + A \]
  perturbs it as follows
  \[ \| (\Delta A, \Delta B) \|_F = O(\epsilon) \]
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The eigenstructure algorithms applied to a BK pencil

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and there exists an equivalent

\[ \|(\Delta M_0, \Delta M_1)\|_F = O(\varepsilon) \]

such that \( \hat{M}_0 := M_0 + \Delta M_0, \hat{M}_1 := M_1 + \Delta M_1 \) and

\[
(I + X) [L(\lambda) + \Delta L(\lambda)] (I + Y) = \\
\begin{bmatrix}
\lambda \hat{M}_1 + \hat{M}_0 & K_{\eta}^T(\lambda) \otimes I_m \\
K_\epsilon(\lambda) \otimes I_n & 0
\end{bmatrix}
\]

with \( \|(X, Y)\|_F = O(\varepsilon) \), which implies that this approach is structurally backward stable if \( \|P(\cdot)\|_F = O(1) \).
Polynomial backward stability

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\|P(\cdot)\|_F^2 := \sum_i \|P_i\|_F^2 = O(1) \implies \|(A, B)\|_F^2 := \|A\|_F^2 + \|B\|_F^2 = O(1)
\]

- The eigenstructure algorithms applied to a BK pencil \(L(\lambda) := \lambda B + A\) perturbs it as follows \(\|(\Delta A, \Delta B)\|_F = O(\varepsilon)\)

- and there exists an equivalent \(\|(\Delta M_0, \Delta M_1)\|_F = O(\varepsilon)\) such that \(\hat{M}_0 := M_0 + \Delta M_0, \hat{M}_1 := M_1 + \Delta M_1\) and

\[
(I + X)[L(\lambda) + \Delta L(\lambda)](I + Y) = \begin{bmatrix}
\lambda \hat{M}_1 + \hat{M}_0 & K_\eta^T(\lambda) \otimes I_m \\
K_\epsilon(\lambda) \otimes I_n & 0
\end{bmatrix}
\]

with \(\|(X, Y)\|_F = O(\varepsilon)\), which implies that this approach is structurally backward stable if \(\|P(\cdot)\|_F = O(1)\)
Polynomial backward stability

Proof of backward stability to a nearby polynomial matrix $P(\lambda) + \Delta P(\lambda)$ provided we scaled the coefficient matrix to 1:

First we restore the anti-triangular structure by strict equivalence

\[
\begin{bmatrix}
I_{(\eta+1)m} & 0 \\
C & I_{\epsilon n}
\end{bmatrix}
\begin{bmatrix}
\frac{\lambda M_1 + M_0}{K_\epsilon(\lambda) \otimes I_n} & K_\eta^T(\lambda) \otimes I_m \\
\lambda \Delta B_{11} + \Delta A_{11} & \lambda \Delta B_{12} + \Delta A_{12} \\
\lambda \Delta B_{21} + \Delta A_{21} & \lambda \Delta B_{22} + \Delta A_{22}
\end{bmatrix}
\begin{bmatrix}
I_{(\epsilon+1)n} & D \\
0 & I_{\eta m}
\end{bmatrix}
= 
\begin{bmatrix}
\frac{\lambda M_1 + M_0}{K_\epsilon(\lambda) \otimes I_n} & K_\eta^T(\lambda) \otimes I_m \\
\frac{\lambda \Delta B_{11} + \Delta A_{11}}{\lambda \Delta B_{21} + \Delta A_{21}} & \frac{\lambda \Delta \tilde{B}_{12} + \Delta \tilde{A}_{12}}{\lambda \Delta \tilde{B}_{22} + \Delta \tilde{A}_{22}}
\end{bmatrix}
\]

This is a nonlinear system of equations $(C, D) = f(A, B, \Delta A, \Delta B)$, but has a solution of norm $\| (C, D) \|_F = O(\epsilon)$. 
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Proof of backward stability to a nearby polynomial matrix $P(\lambda) + \Delta P(\lambda)$ provided we scaled the coefficient matrix to 1:

First we restore the anti-triangular structure by strict equivalence

$$\begin{bmatrix} I_{(\eta+1)m} & 0 \\ C & I_{\epsilon n} \end{bmatrix} \left( \begin{bmatrix} \lambda M_1 + M_0 & K_\eta^T(\lambda) \otimes I_m \\ K_\epsilon(\lambda) \otimes I_n & 0 \end{bmatrix} \right) + \begin{bmatrix} \lambda \Delta B_{11} + \Delta A_{11} & \lambda \Delta B_{12} + \Delta A_{12} \\ \lambda \Delta B_{21} + \Delta A_{21} & \lambda \Delta B_{22} + \Delta A_{22} \end{bmatrix} \begin{bmatrix} I_{(\epsilon+1)n} & D \\ 0 & I_{\eta m} \end{bmatrix} = \begin{bmatrix} \lambda M_1 + M_0 & K_\eta^T(\lambda) \otimes I_m \\ K_\epsilon(\lambda) \otimes I_n & 0 \end{bmatrix} + \begin{bmatrix} \lambda \Delta B_{11} + \Delta A_{11} & \lambda \Delta \tilde{B}_{12} + \Delta \tilde{A}_{12} \\ \lambda \Delta \tilde{B}_{21} + \Delta \tilde{A}_{21} & 0 \end{bmatrix}$$

This is a nonlinear system of equations $(C, D) = f(A, B, \Delta A, \Delta B)$, but has a solution of norm $\| (C, D) \|_F = O(\epsilon)$. 
Polynomial backward stability

Second, we “restore” the perturbed dual minimal bases

\[
(I + X_\epsilon) \left( K_\epsilon(\lambda) \otimes I_n + \lambda \tilde{B}_{21} + \Delta \tilde{A}_{21} \right) (I + Y_\epsilon) = K_\epsilon(\lambda) \otimes I_n
\]

\[
(I + Y_\eta^T) \left( K_\eta(\lambda) \otimes I_m + \lambda \tilde{B}_{12}^T + \Delta \tilde{A}_{12}^T \right) (I + X_\eta^T) = K_\eta(\lambda) \otimes I_m
\]

Bounding \(\|(C, D)\|_F, \|(X_\epsilon, Y_\epsilon)\|_F\) and \(\|(X_\eta, Y_\eta)\|_F\), we can prove the theorem.

Let \(A + \lambda B\) be a block Kronecker linearization of a polynomial matrix \(P(\lambda)\) and let \(\Delta A + \lambda \Delta B\) be the backward error induced by the eigenstructure algorithm. Then the corresponding backward error \(\Delta P(\cdot)\) has a norm satisfying

\[
\|\Delta P(\cdot)\|_F \leq d^2 \sqrt{m + n} \|\Delta A, \Delta B\|_F = O(\epsilon)
\]
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Bounding \(\| (C, D)\|_F, \| (X_\epsilon, Y_\epsilon)\|_F\) and \(\| (X_\eta, Y_\eta)\|_F\), we can prove

**Theorem**

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\[\| \Delta P(\cdot)\|_F \leq d^2 \sqrt{m + n}\| (\Delta A, \Delta B)\|_F = O(\epsilon)\]
The rational case

It follows from realization theory that the “zero structure” of a rational matrix

\[ R(\lambda) := P(\lambda) + C(\lambda I_\ell - A)^{-1} B, \]

where

\[ P(\lambda) := \lambda^d P_d + \lambda^{d-1} P_{d-1} + \cdots + \lambda P_1 + P_0 \]

is the same as the “zero structure” of the so-called system matrix

\[ S(\lambda) := \begin{bmatrix} P(\lambda) & C \\ B & A - \lambda I_\ell \end{bmatrix} \]

provided \((A, B)\) is controllable and \((A, C)\) is observable.

Proof: Schur complement \(\implies\) \(\det S(\lambda) = \det(A - \lambda I_\ell) \cdot \det R(\lambda)\)
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Now linearize $S(\lambda)$ [ADMZ2018]

The rational matrix

$$R(\lambda) := P(\lambda) + C(\lambda I_\ell - A)^{-1}B,$$

has the same “zero structure” as

$$L(\lambda) := \begin{bmatrix} M(\lambda) & N_\eta^T C & K_\eta^T(\lambda) \\ BN_\epsilon & A - \lambda I_\ell & 0 \\ K_\epsilon(\lambda) & 0 & 0 \end{bmatrix}.$$

where

$$\begin{bmatrix} K_k(\lambda) \\ N_k \end{bmatrix} = \begin{bmatrix} I & -\lambda I \\ \vdots & \vdots \\ 0 & I - \lambda I \end{bmatrix}.$$
Proof [ADMZ2018]

There exist unimodular transformations $V_\epsilon(\lambda)$ and $V_\eta(\lambda)$ such that

\[
\begin{bmatrix}
V_\eta^T(\lambda) & 0 \\
0 & I_{\epsilon n}
\end{bmatrix}
L(\lambda)
\begin{bmatrix}
V_\epsilon(\lambda) & 0 \\
0 & I_{\eta m}
\end{bmatrix}
= 
\begin{bmatrix}
X(\lambda) & Y(\lambda) & 0 & I_{\eta m} \\
Z(\lambda) & P(\lambda) & C & 0 \\
0 & B & A - \lambda I_\ell & 0 \\
I_{\epsilon n} & 0 & 0 & 0
\end{bmatrix}
\]

which implies that $L(\lambda)$ has the same zero structure as

\[
S(\lambda) := \begin{bmatrix}
P(\lambda) & C \\
B & A - \lambda I_\ell
\end{bmatrix}
\]

and hence also as $R(\lambda)$.

Clearly, we can use the same construction as before for $\lambda M_1 - M_0$. 
Proof [ADMZ2018]

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0 & I_{\ell n}
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V_\epsilon(\lambda) & 0 \\
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X(\lambda) & Y(\lambda) & 0 & I_{\eta m} \\
Z(\lambda) & P(\lambda) & C & 0 \\
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Example

No operations are needed: the $m \times n$ rational matrix

$$R(\lambda) := \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0 + C(\lambda I_\ell - A)^{-1} B$$

has the same “zero structure” as

$$L(\lambda) := \begin{bmatrix} \lambda P_3 + P_2 & 0 & 0 & I_m \\ 0 & \lambda P_1 + P_0 & C & -\lambda I_m \\ 0 & B & A - \lambda I_\ell & 0 \\ I_n & -\lambda I_n & 0 & 0 \end{bmatrix}.$$  

Notice that symmetry in $R(\lambda)$ is inherited by $L(\lambda)$. 
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Structured backward stability [DQV2018]

Applying a stable algorithm to

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L(\lambda) := \begin{bmatrix}
M(\lambda) & N_{\eta}^T C & K_{\eta}^T (\lambda) \\
BN_{\epsilon} & A - \lambda I_{\ell} & 0 \\
K_{\epsilon}(\lambda) & 0 & 0
\end{bmatrix}
\]

will introduce an error \( \Delta L(\lambda) \) that destroys its structure but it can be restored by equivalence transformations

\[
(I + X) [L(\lambda) + \Delta L(\lambda)] (I + Y) = 
\begin{bmatrix}
\hat{M}(\lambda) & N_{\eta}^T \hat{C} & K_{\eta}^T (\lambda) \\
\hat{B}N_{\epsilon} & \hat{A} - \lambda I_{\ell} & 0 \\
K_{\epsilon}(\lambda) & 0 & 0
\end{bmatrix}
\]

where \( \|(X, Y)\|_F \leq p \left[ d, \|A\|_{2, \max(\epsilon, \eta)}^2, \|B\|_2, \|C\|_2 \right] \|\Delta L(\cdot)\|_F \)
Structured backward stability [DQV2018]

Applying a stable algorithm to

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L(\lambda) := \begin{bmatrix}
M(\lambda) & N^T_\eta C & K^T_\eta(\lambda) \\
BN_\epsilon & A - \lambda I_\ell & 0 \\
K_\epsilon(\lambda) & 0 & 0
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K_\epsilon(\lambda) & 0 & 0
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\]

where \( \|(X, Y)\|_F \leq p \left[ d, \|A\|_{\max(\epsilon, \eta)}, \|B\|_2, \|C\|_2 \right] \|\Delta L(.)\|_F \)
Scaling of $R(\lambda)$ [DQV2018]

There exists a scaling $R_s(\lambda_s) := d_r R(d_\lambda \lambda)$ which guarantees that

$$\|D_{ata}\|_F = \mathcal{O}(1), \quad \text{where} \quad D_{ata} = (A, B, C, P_0, \ldots, P_d)$$

Then $\|L(.)\|_F = \mathcal{O}(1)$, $\|\Delta L(.)\|_F = \mathcal{O}(\varepsilon)$ and $\|\Delta D_{ata}\|_F = \mathcal{O}(\varepsilon)$
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Then $\|L(.)\|_F = O(1)$, $\|\Delta L(.)\|_F = O(\varepsilon)$ and $\|\Delta D_{ata}\|_F = O(\varepsilon)$
Conclusions

- We derived Block Kronecker linearizations of polynomial and rational matrices
  - The proof is based on simple unimodular transformations and dual minimal bases
- We can scale the linearizations so that they are “balanced”
- We have then “polynomial and rational backward stability” for any stable eigenstructure algorithm
- We can preserve certain symmetries
- We can extend this to degree $\ell$ block Kronecker polynomial matrices
- We can extend this to other bases than monomial ones
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Block Kronecker ℓ-ifications

For ℓ = 2 we can obtain ”quadratifications” of an even polynomial matrix as follows

\[ P(\lambda) = \lambda^6 P_6 + \lambda^5 P_5 + \lambda^4 P_4 + \lambda^3 P_3 + \lambda^2 P_2 + \lambda P_1 + P_0. \]

\[ \lambda^2 C + \lambda B + A = \begin{bmatrix} \lambda^2 P_6 + \lambda P_5 + P_4 & \lambda P_3/2 & I_m \\ \lambda P_3/2 & \lambda^2 P_2 + \lambda P_1 + P_0 & -\lambda^2 I_m \\ I_n & -\lambda^2 I_n & 0 \end{bmatrix}. \]

This is also a symmetric ”quadratification” if \( P(\lambda) \) is symmetric.

Proof is very analogous.
Block Kronecker $\ell$-ifications

For $\ell = 2$ we can obtain "quadratifications" of an even polynomial matrix as follows

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Similar developments also exist for other basis functions (Chebyshev, Lagrange, barycentric ...).
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References


- F. Dopico, J. Perez and P. Van Dooren, *Structured backward error analysis of linearized structured polynomial eigenvalue problems*, Math Comp, 2018

