University of Cagliari, Italy

Recovering the electrical conductivity of the soil via linear integral equations

Patricia Díaz de Alba

In coll. with **Luisa Fermo**, **Cornelis van der Mee** and **Giuseppe Rodriguez**

NASCA18, Kalamata
July 2-6, 2018

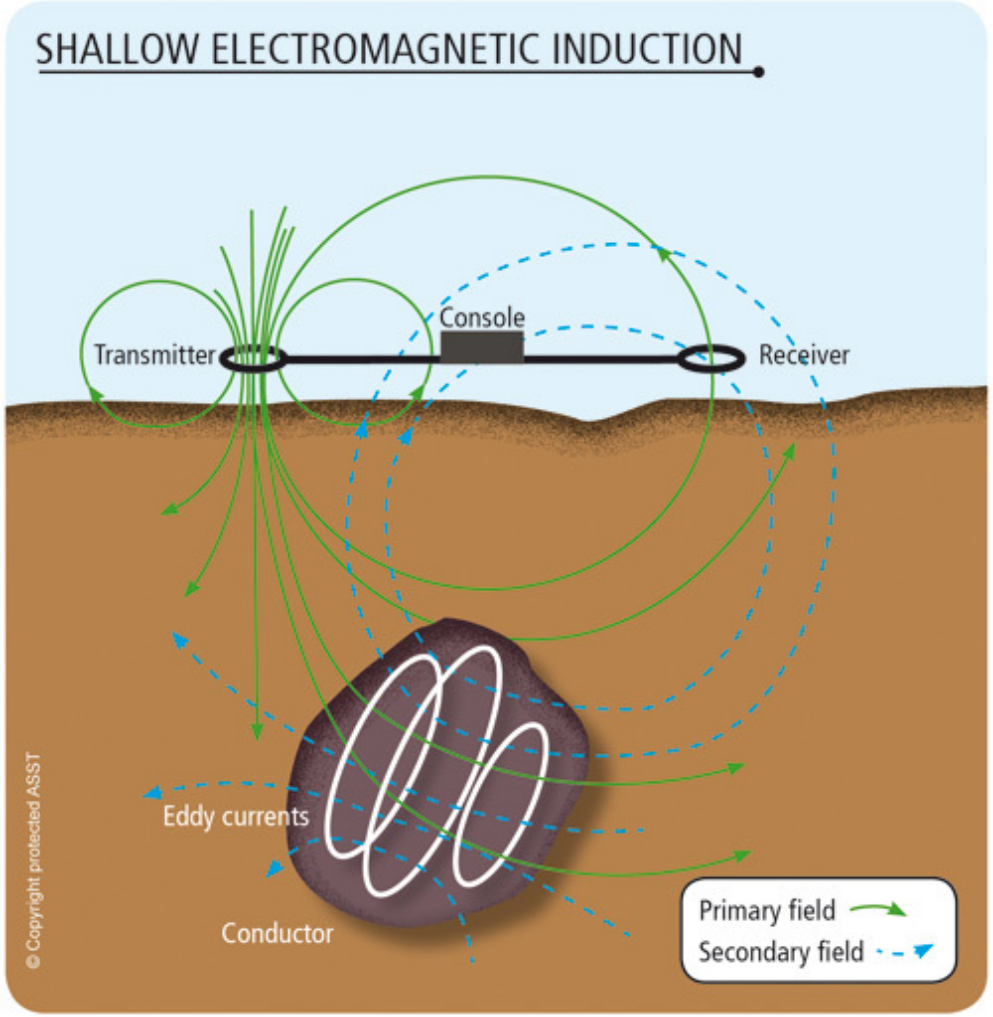
Motivation: Reconstructing the electrical conductivity of the soil by *Electromagnetic induction* (EMI) techniques.

Main applications

- Soil studies.
- Hydrological and hydrogeological characterizations.
- Hazardous waste characterization studies.
- Precision-agriculture applications.
- Archaeological surveys.
- Geotechnical investigations.
- Unexploded ordnance (UXO) detection.

The main device used in applied Geophysics is a Ground Conductivity Meter.

How does it work? $\Rightarrow$

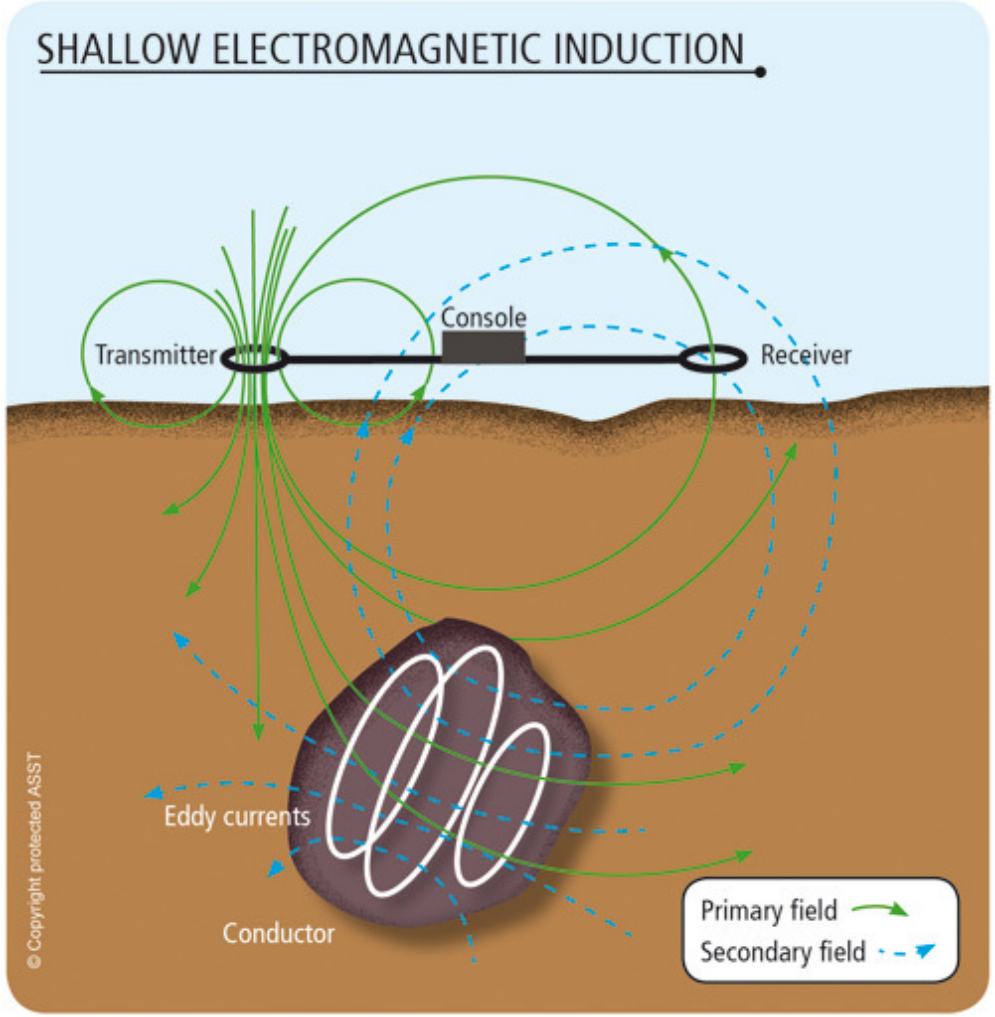


The main device used in applied Geophysics is a Ground Conductivity Meter.

The following parameters

- orientation (vert/horiz)
- height h over the ground
- angular frequency $\omega = 2\pi f$
- inter-coil distance ρ

can be varied in order to generate multiple measurements for each geographical position and realize data inversion, that is, approximate $\sigma(z)$.



McNeill developed in 1980 the following linear model for the Geonics EM-38 ($\rho = 1m$; $f = 14600$ Hz),

$$\begin{cases} \displaystyle\int_0^\infty k^V(z+h)\,\sigma(z)\,dz = g^V(h), & 0 \le h < \infty, \\ \\ \displaystyle\int_0^\infty k^H(z+h)\,\sigma(z)\,dz = g^H(h), & 0 \le h < \infty, \end{cases}$$

where

$$k^V(z) = \frac{4z}{(4z^2+1)^{3/2}}, \qquad k^H(z) = 2 - \frac{4z}{(4z^2+1)^{1/2}}.$$

J. D. McNeill
Electromagnetic terrain conductivity measurement at low induction numbers.
Technical Report TN-6 Geonics Limited, 1980.

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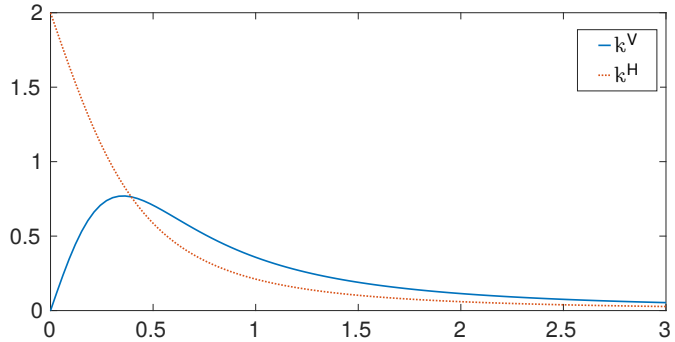
- $\sigma(z)$ is the electrical conductivity of the soil at deph z.
- $g^V(h)$ and $g^H(h)$ represent the apparent conductivity of the soil sensed by the GCM at height h above the ground, in correspondence to the two possible orientations of the coils.



On the uniqueness of the solution

Theorem

The operators

$$K^V, K^H : C^0([0,\infty)) \cap L^\infty([0,\infty)) \to C^0([0,\infty)) \cap L^\infty([0,\infty))$$

defined as

$$K^V(h) = \int_0^\infty k^V(z+h)\,\sigma(z)\,dz, \qquad K^H(h) = \int_0^\infty k^H(z+h)\,\sigma(z)\,dz$$

have a trivial null space.

On the uniqueness of the solution

Theorem

The null space of the operators

$$K^V, K^H : L^p([0,\infty)) \cap L^\infty([0,\infty)) \to L^p([0,\infty)) \cap L^\infty([0,\infty))$$

with $p \geq 1$ *defined as*

$$K^V(h) = \int_0^\infty k^V(z+h)\,\sigma(z)\,dz, \qquad K^H(h) = \int_0^\infty k^H(z+h)\,\sigma(z)\,dz,$$

consists of the functions which vanish everywhere, except on a set of measure zero.

On the existence of the solution

The solution have the following form

$$\sigma = \sum_{j=1}^{\infty} \frac{\langle g, \varphi_j \rangle}{\lambda_j} \varphi_j,$$

where

- $\{\varphi_j\}_{j=1}^{\infty} \in L^2([0,\infty))$ are the eigenfunctions of K^J such that $K^J \varphi_j = \lambda_j \varphi_j$ with $\lambda_j \neq 0$;
- $\langle g, \varphi_j \rangle \in L^2([0, \infty))$.

The series converges only if the right-hand side g fullfills the Picard condition

$$\sum_{j=1}^{\infty} \frac{\langle g, \varphi_j \rangle^2}{\lambda_j^2} < \infty.$$

In applications, $|\sigma(z)| \leq C$ in the interval $[\zeta, \infty)$ with $\zeta > 0$. Assuming that we know C,

$$\begin{cases} \displaystyle\int_0^\zeta k^V(z+h)\,\sigma(z)\,dz = \tilde{g}^V(h), & 0 \leq h < \infty, \\ \displaystyle\int_0^\zeta k^H(z+h)\,\sigma(z)\,dz = \tilde{g}^H(h), & 0 \leq h < \infty, \end{cases}$$

where

$$\tilde{g}^{V,H} = \begin{cases} g^V(h) - \dfrac{C}{\sqrt{4(h+\zeta)^2+1}}, \\ g^H(h) - C\left(\sqrt{4(h+\zeta)^2+1} - 2(h+\zeta)\right). \end{cases}$$

Numerical treatment

Approximation of σ:

1. By using a piecewise constant function.
2. By using a linear spline.
3. By using a Bernstein polynomials.

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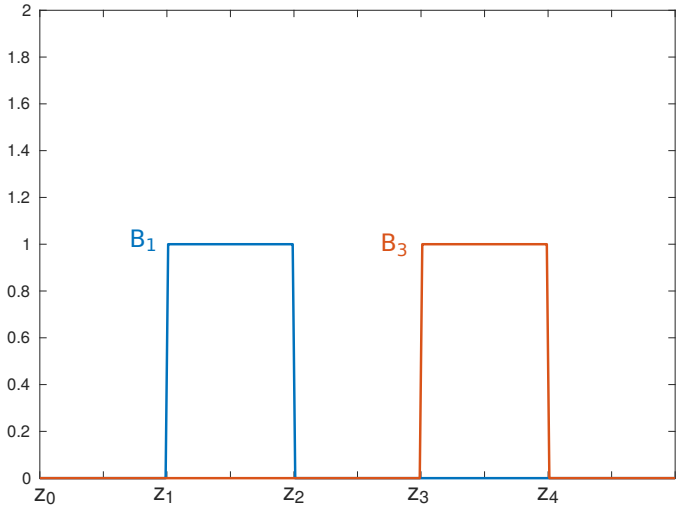
Case 1: We represent $\sigma(z)$ by means of B-splines of order 1 as

$$\sigma(z) \simeq \sum_{j=1}^{n} \alpha_j \; B_j(z),$$

with

$$B_j(z) = \begin{cases} 1, & z_j \leq z < z_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$



By replacing the expression of σ and collocating at h_i, we get for the vertical orientation

$$\sum_{j=0}^{n} \alpha_j \int_0^{\zeta} \phi^V(z+h_i)\ B_j(z)\ dz = g^V(h_i), \quad i = 0, \ldots, m-1.$$

Taking into account

$$\int_0^{\zeta} \phi^V(z+h_i) B_j(z) dz = \int_{z_j}^{z_{j+1}} \phi^V(h_i+z)\, dz$$
$$= \frac{1}{\sqrt{4(h_i+z_j)^2+1}} - \frac{1}{\sqrt{4(h_i+z_{j+1})^2+1}} := \phi_{ij}^V$$

we get

$$\sum_{j=0}^{n} \alpha_j\ \phi_{ij}^V = \tilde{g}_i^V, \qquad \tilde{g}^V(h_i) = \tilde{g}_i^V.$$

Remark: Similar expression holds for the horizontal orientation.

Setting $b_i^V = \tilde{g}^V(h_i)$ and $b_i^H = \tilde{g}^H(h_i)$, we can consider the linear system

$$\mathbf{\Phi}\boldsymbol{\alpha} = \mathbf{b}$$

where $\boldsymbol{\alpha} = [\alpha_j]_{j=0,\ldots,n}$ is the array of the unknowns, $\mathbf{\Phi}$ is the matrix of coefficients defined as

$$\mathbf{\Phi} = \begin{bmatrix} \boldsymbol{\phi}^V \\ \boldsymbol{\phi}^H \end{bmatrix}, \quad \boldsymbol{\phi}^V = [\phi_{ij}^V]_{\substack{i=1,\ldots,m,\\ j=1,\ldots,n}}, \quad \boldsymbol{\phi}^H = [\phi_{ij}^H]_{\substack{i=1,\ldots,m,\\ j=1,\ldots,n}},$$

and $\mathbf{b}$ is the array of the right–hand side

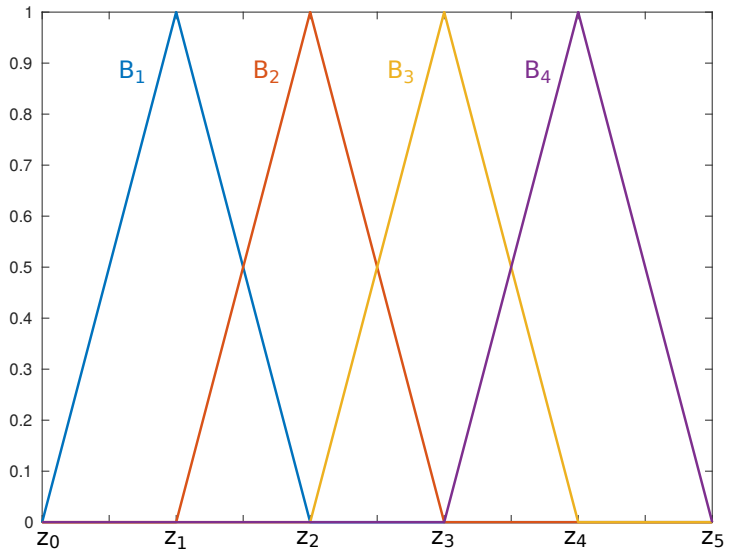
$$\mathbf{b} = \begin{bmatrix} \mathbf{b}^V \\ \mathbf{b}^H \end{bmatrix}, \quad \mathbf{b}^V = [b_1^V, \ldots, b_m^V]^T, \quad \mathbf{b}^H = [b_1^H, \ldots, b_m^H]^T.$$

Case 2: We represent $\sigma(z)$ by using a linear combination of linear B-splines,

$$\sigma(z) \simeq \sum_{j=0}^{n} \alpha_j \; B_j(z),$$

where

$$B_j(z) = \begin{cases} \dfrac{1}{\delta_j}(z - z_{j-1}), & z_{j-1} \leq z < z_j, \\ \dfrac{1}{\delta_{j+1}}(z_{j+1} - z), & z_j \leq z < z_{j+1}, \\ 0, & \text{otherwise}, \end{cases}$$



with

$$B_0(z) = \begin{cases} \dfrac{1}{\delta_1}(z_1 - z), & z_0 \leq z < z_1, \\ 0, & \text{otherwise}, \end{cases}$$

and $\delta_j = z_j - z_{j-1}$ and $j = 0, 1, \ldots$.

By a similar procedure, we get a matrix system

$$\mathbf{\Phi}\boldsymbol{\alpha} = \mathbf{b}$$

where

$$\phi_{ij}^V = \begin{cases} \dfrac{1}{\sqrt{4(h_i+z_0)^2+1}} - \dfrac{1}{\delta_1}\psi^V(h_i+z_0, h_i+z_1), & \text{if } j=0 \\ \dfrac{1}{\delta_j}\psi^V(h_i+z_{j-1}, h_i+z_j) - \dfrac{1}{\delta_{j+1}}\psi^V(h_i+z_j, h_i+z_{j+1}), & \text{if } j \geq 1, \end{cases}$$

with

$$\psi^V(a,b) = \frac{\sinh^{-1}(2b) - \sinh^{-1}(2a)}{2},$$

Remark: Similar expression holds for the horizontal orientation.

Case 3: We introduce the change of variable $y = z/\zeta$, which leads to

$$\zeta \int_0^1 k^V(\zeta y + h)\sigma(\zeta y)\, dy = \tilde{g}^V(h).$$

Now, we approximate the integrand by a Bernstein polynomial

$$k^V(\zeta y + h)\sigma(\zeta y) = \sum_{j=0}^{n} k^V(\zeta y_j + h)\sigma(\zeta y_j)\, p_{nj}(y),$$

where $y_j = j/n$, $j = 0, 1, \ldots, n$, and

$$p_{nj} = \binom{n}{j} z^j (1-z)^{n-j}.$$

By substituting, and collocating the resulting equation at the heights h_i, $i = 1, \ldots, m$, we obtain

$$\sum_{j=0}^{n} k^V(\zeta y_j + h_i) d_j \sigma(\zeta y_j) = \tilde{g}^V(h_i), \quad i = 1, \ldots, m,$$

where

$$d_j = \zeta \int_0^1 p_{n,j}(y)\, dy = \zeta \binom{n}{j} B(j+1, n-j+1) = \frac{\zeta}{n+1},$$

being $B(x, y)$ the beta function.

Then, by setting

$$\phi_{ij}^V = \frac{\zeta}{n+1} k^V(\zeta y_j + h_i), \quad \sigma_j = \sigma(\zeta y_j), \quad b_i^V = \tilde{g}^V(h_i),$$

and repeating the same procedure for the second integral equation, we get the linear system

$$\Phi \boldsymbol{\sigma} = \mathbf{b}.$$

Regularization

A way to treat the ill–conditioning of $\mathbf{\Phi}$ is to derive a new problem with a well–conditioned rank–deficient coefficient matrix.

The best rank k approximation $\mathbf{\Phi}_k$, according to the 2-norm, can be obtained by the SVD decomposition, $\mathbf{\Phi} = U\Sigma V^T$. The truncated SVD regularization solves the problem

$$\begin{cases} \min\limits_{\boldsymbol{\sigma}} \|\boldsymbol{\sigma}\|, \\ \text{subj. to } \min\limits_{\boldsymbol{\sigma}} \|\Phi\boldsymbol{\sigma} - \mathbf{b}\|, \end{cases}$$

and the minimal norm solution can be written as

$$\boldsymbol{\sigma}_k = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i,$$

where k is the truncation parameter.

Regularization

Let us introduce a regularization matrix $L \in \mathbb{R}^{p \times n}$ $(p \leq n)$.
Common choices are $L = D_1, D_2$.

The minimal L-norm solution is the vector which solves

$$\begin{cases} \min\limits_{\boldsymbol{\sigma}} \|L\boldsymbol{\sigma}\|, \\ \text{subj. to } \min\limits_{\boldsymbol{\sigma}} \|\Phi\boldsymbol{\sigma} - \mathbf{b}\|, \end{cases}$$

under the assumption $\mathcal{N}(\Phi) \cap \mathcal{N}(L) = \{0\}$.
By the generalized singular value decomposition (GSVD) of (Φ, L)

$$\Phi = U \begin{bmatrix} \Sigma & 0 \\ 0 & I_{n-p} \end{bmatrix} Z^{-1}, \quad L = V \begin{bmatrix} M & 0 \end{bmatrix} Z^{-1},$$

we compute the truncated GSVD (TGSVD) solution $\boldsymbol{\sigma}_k$, where $k \leq p$ is the regularization parameter

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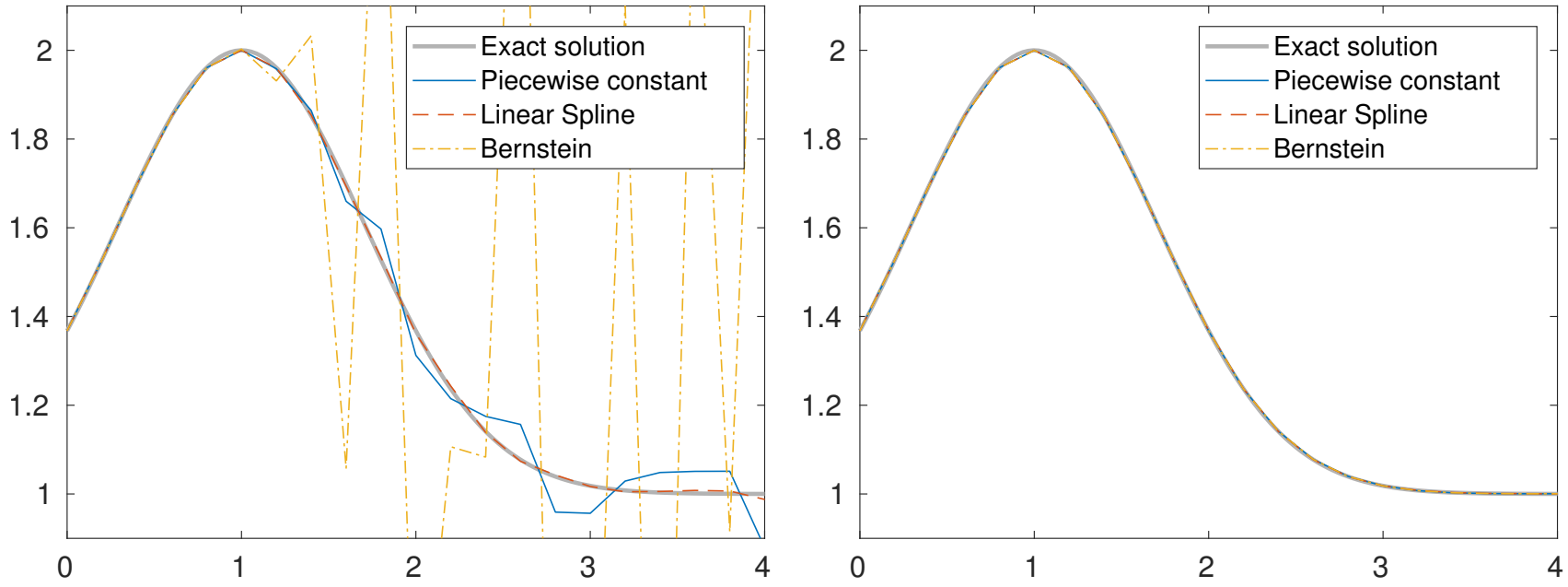
we compute the truncated GSVD (TGSVD) solution

$$\boldsymbol{\sigma}_k = \sum_{i=p-k+1}^{p} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{z}_i + \sum_{i=p+1}^{n} (\mathbf{u}_i^T \mathbf{b}) \mathbf{z}_i,$$

The choice of the reg. parameter k is crucial: $k^* = \arg\min_k \|\boldsymbol{\sigma}_k - \boldsymbol{\sigma}\|$

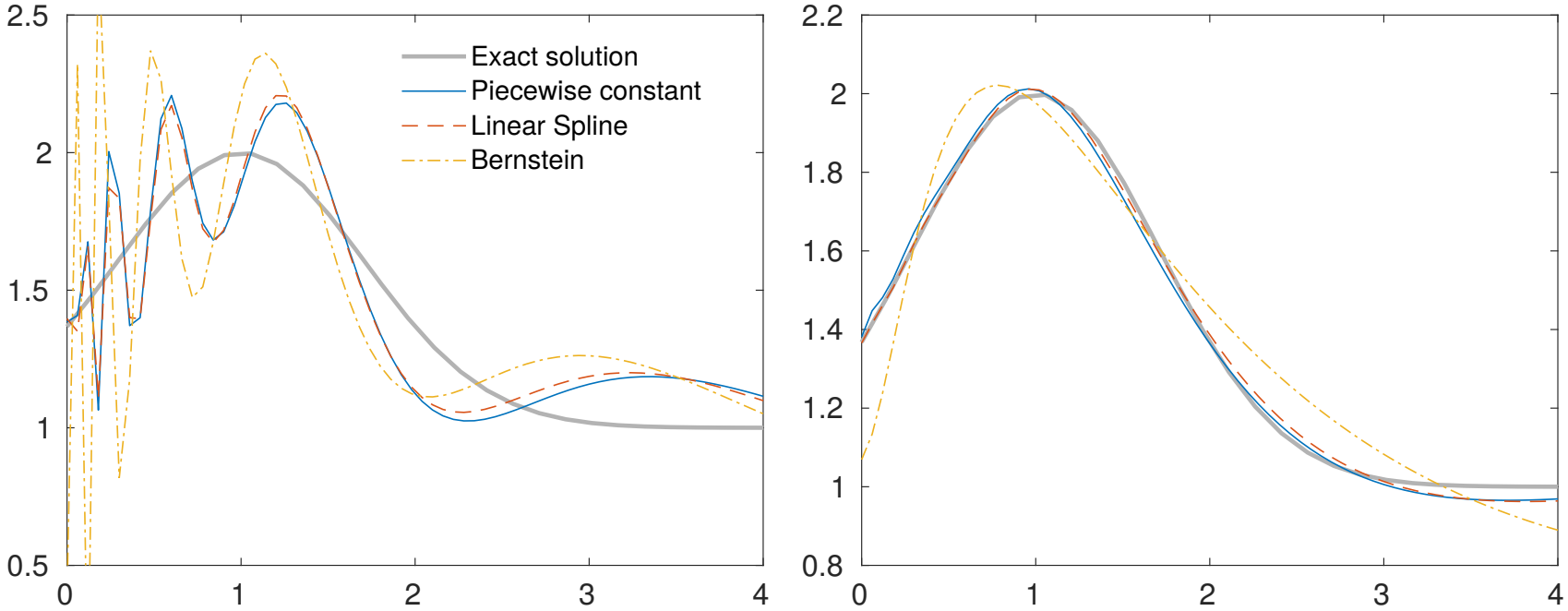
Effectivenes of regularization



$$m = 20;\ n = 30;\ \eta = 0;\ \zeta = 6;\ \sigma^a(z) = e^{-(z-1)^2} + 1$$

left: no regularization; right: TSVD regularization

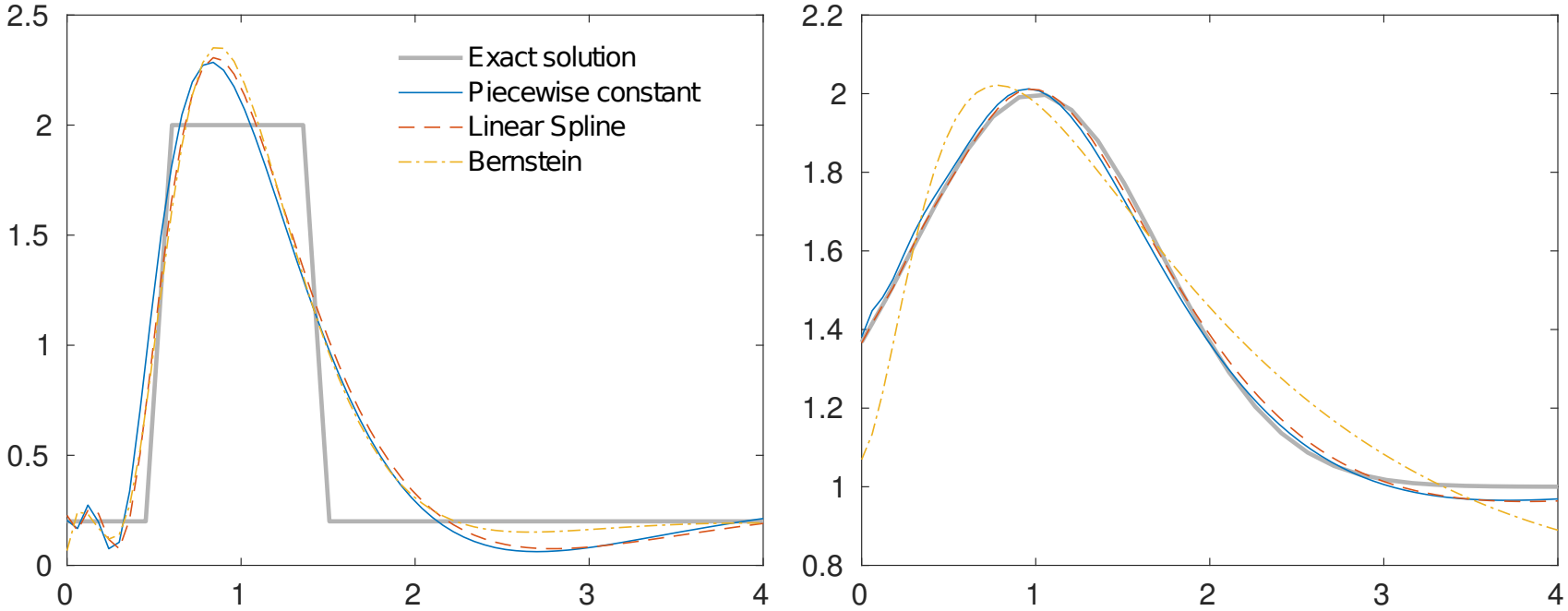
Comparison between $L = I$ and $L = D_1$



$m = 20$; $n = 500$; $\eta = 0$; $\zeta = 30$, $\sigma^a(z) = e^{-(z-1)^2} + 1$

left: $L = I$; right: $L = D_1$

Comparison between tow dirrefent profiles of the conductivity

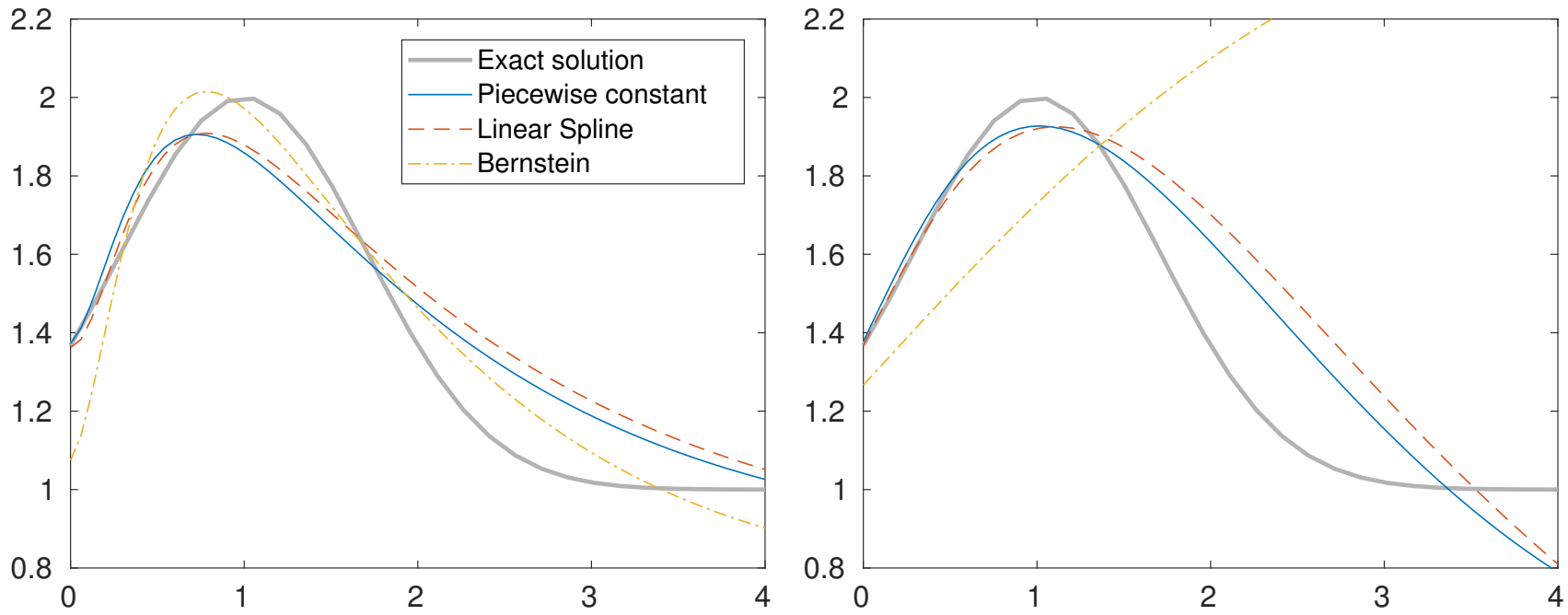


$$m = 20;\ n = 500;\ \eta = 0;\ \zeta = 30$$

left: $\sigma^b(z) = \begin{cases} 0.2, & z \in (-\infty,\, 0.5), \\ 2, & z \in [0.5,\, 1.5], \\ 0.2, & z \in (1.5,\, \infty). \end{cases}$;

right: $\sigma^a(z) = e^{-(z-1)^2} + 1$

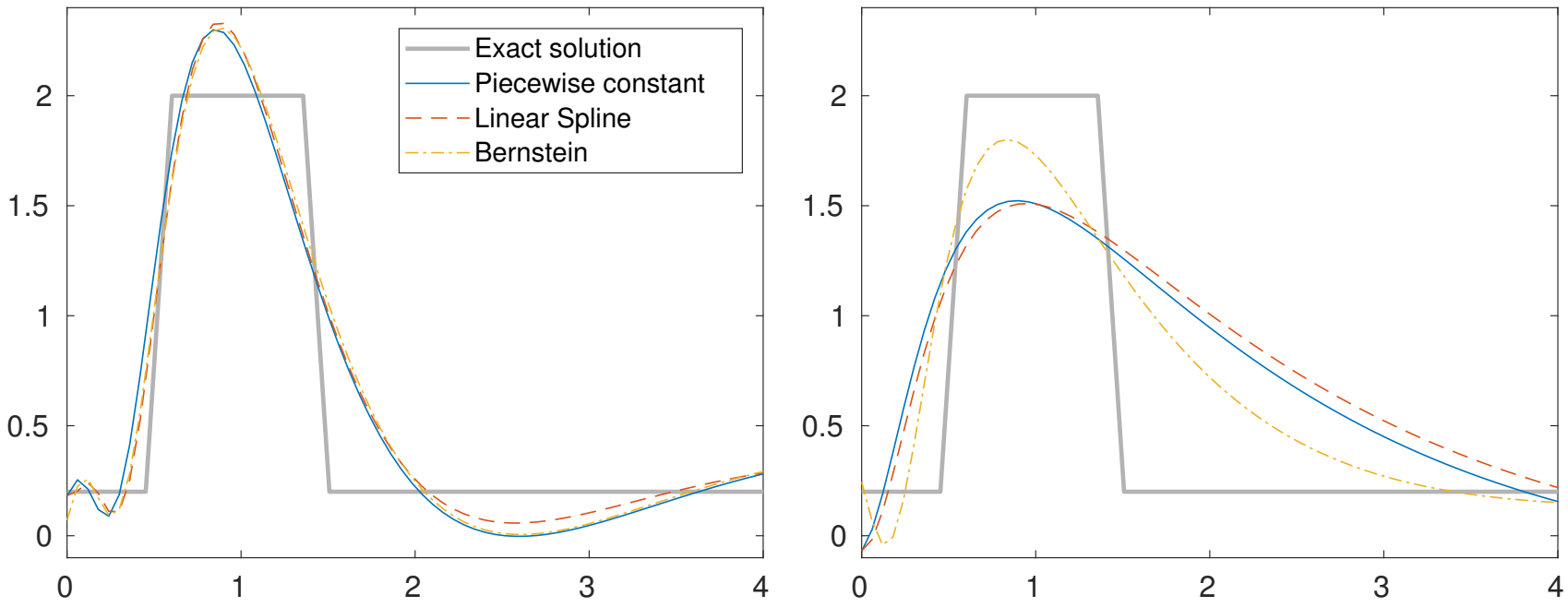
Different regularization matrix



$$m = 20;\ n = 500;\ \eta = 10^{-3};\ \zeta = 30;\ \sigma^a(z) = e^{-(z-1)^2} + 1$$

left: $L = D_1$; right: $L = D_2$

Different noise levels



$$m = 20;\ n = 500;\ L = D_1;\ \zeta = 30;\ \sigma^b(z) = \begin{cases} 0.2, & z \in (-\infty,\, 0.5), \\ 2, & z \in [0.5,\, 1.5], \\ 0.2, & z \in (1.5,\, \infty). \end{cases}$$

left: $\eta = 10^{-6}$; right: $\eta = 10^{-3}$

THANK YOU FOR YOUR ATTENTION!