Recovering the electrical conductivity of the soil via linear integral equations

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Motivation: Reconstructing the electrical conductivity of the soil by Electromagnetic induction (EMI) techniques.

Main applications

- Soil studies.
- Hydrological and hydrogeological characterizations.
- Hazardous waste characterization studies.
- Precision-agriculture applications.
- Archaeological surveys.
- Geotechnical investigations.
- Unexploded ordnance (UXO) detection.
The main device used in applied Geophysics is a **Ground Conductivity Meter**.

How does it work? ⇒

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**SHALLOW ELECTROMAGNETIC INDUCTION**

- **Transmitter**
- **Console**
- **Receiver**
- **Eddy currents**
- **Conductor**
- **Primary field**
- **Secondary field**

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The main device used in applied Geophysics is a **Ground Conductivity Meter**.

The following parameters can be varied in order to generate multiple measurements for each geographical position and realize data inversion, that is, approximate $\sigma(z)$.

- orientation (vert/horiz)
- height $h$ over the ground
- angular frequency $\omega = 2\pi f$
- inter-coil distance $\rho$
McNeill developed in 1980 the following linear model for the Geonics EM-38 ($\rho = 1 m; f = 14600 \text{ Hz}$),

$$
\left\{
\begin{aligned}
\int_{0}^{\infty} k^V(z + h) \sigma(z) \, dz &= g^V(h), & 0 \leq h < \infty, \\
\int_{0}^{\infty} k^H(z + h) \sigma(z) \, dz &= g^H(h), & 0 \leq h < \infty,
\end{aligned}
\right.
$$

where

$$
k^V(z) = \frac{4z}{(4z^2 + 1)^{3/2}}, \quad k^H(z) = 2 - \frac{4z}{(4z^2 + 1)^{1/2}}.
$$

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J. D. McNeill

McNeill developed in 1980 the following linear model for the Geonics EM-38 ($\rho = 1m; f = 14600$ Hz),

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\end{align*}
\]

where

\[
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\]

- $\sigma(z)$ is the electrical conductivity of the soil at depth $z$.
- $g^V(h)$ and $g^H(h)$ represent the apparent conductivity of the soil sensed by the GCM at height $h$ above the ground, in correspondence to the two possible orientations of the coils.
Theorem

The operators

\[ K^V, K^H : C^0([0, \infty)) \cap L^\infty([0, \infty)) \rightarrow C^0([0, \infty)) \cap L^\infty([0, \infty)) \]

defined as

\[ K^V(h) = \int_0^\infty k^V(z + h) \sigma(z) \, dz, \quad K^H(h) = \int_0^\infty k^H(z + h) \sigma(z) \, dz \]

have a trivial null space.
On the uniqueness of the solution

**Theorem**

The null space of the operators

\[ K^V, K^H : L^p([0, \infty)) \cap L^\infty([0, \infty)) \to L^p([0, \infty)) \cap L^\infty([0, \infty)) \]

with \( p \geq 1 \) defined as

\[
K^V(h) = \int_0^\infty k^V(z + h) \sigma(z) \, dz, \quad K^H(h) = \int_0^\infty k^H(z + h) \sigma(z) \, dz,
\]

consists of the functions which vanish everywhere, except on a set of measure zero.
On the existence of the solution

The solution have the following form

$$\sigma = \sum_{j=1}^{\infty} \frac{\langle g, \varphi_j \rangle}{\lambda_j} \varphi_j,$$

where

- \( \{ \varphi_j \}_{j=1}^{\infty} \in L^2([0, \infty)) \) are the eigenfunctions of \( K^J \) such that \( K^J \varphi_j = \lambda_j \varphi_j \) with \( \lambda_j \neq 0 \);
- \( \langle g, \varphi_j \rangle \in L^2([0, \infty)) \).

The series converges only if the right-hand side \( g \) fullfills the Picard condition

$$\sum_{j=1}^{\infty} \frac{\langle g, \varphi_j \rangle^2}{\lambda_j^2} < \infty.$$
In applications, $|\sigma(z)| \leq C$ in the interval $[\zeta, \infty)$ with $\zeta > 0$. Assuming that we know $C$, we have

\[
\begin{align*}
\int_0^\zeta k^V(z+h)\sigma(z)\,dz &= \tilde{g}^V(h), & 0 \leq h < \infty, \\
\int_0^\zeta k^H(z+h)\sigma(z)\,dz &= \tilde{g}^H(h), & 0 \leq h < \infty,
\end{align*}
\]

where

\[
\tilde{g}^{V,H} = \begin{cases} 
g^V(h) - \frac{C}{\sqrt{4(h+\zeta)^2 + 1}}, \\
g^H(h) - C \left( \sqrt{4(h+\zeta)^2 + 1} - 2(h+\zeta) \right). \end{cases}
\]
Numerical treatment

Approximation of $\sigma$:

1. By using a piecewise constant function.
2. By using a linear spline.
3. By using a Bernstein polynomials.
Numerical treatment

Approximation of $\sigma$:

1. By using a piecewise constant function.
2. By using a linear spline.
3. By using a Bernstein polynomials.

**Case 1**: We represent $\sigma(z)$ by means of B-splines of order 1 as

$$\sigma(z) \simeq \sum_{j=1}^{n} \alpha_j B_j(z),$$

with

$$B_j(z) = \begin{cases} 
1, & z_j \leq z < z_{j+1}, \\
0, & \text{otherwise.}
\end{cases}$$
By replacing the expression of $\sigma$ and collocating at $h_i$, we get for the vertical orientation

$$
\sum_{j=0}^{n} \alpha_j \int_{0}^{\zeta} \phi_V(z + h_i) B_j(z) \, dz = g^V(h_i), \quad i = 0, \ldots, m - 1.
$$

Taking into account

$$
\int_{0}^{\zeta} \phi_V(z + h_i) B_j(z) \, dz = \int_{z_j}^{z_j+1} \phi_V(h_i + z) \, dz
$$

$$
= \frac{1}{\sqrt{4(h_i + z_j)^2 + 1}} - \frac{1}{\sqrt{4(h_i + z_{j+1})^2 + 1}} := \phi_{ij}^V
$$

we get

$$
\sum_{j=0}^{n} \alpha_j \phi_{ij}^V = \tilde{g}_i^V, \quad \tilde{g}^V(h_i) = \tilde{g}_i^V.
$$

**Remark:** Similar expression holds for the horizontal orientation.
Setting \( b^V_i = \tilde{g}^V(h_i) \) and \( b^H_i = \tilde{g}^H(h_i) \), we can consider the linear system

\[
\Phi \alpha = b
\]

where \( \alpha = [\alpha_j]_{j=0,...,n} \) is the array of the unknowns, \( \Phi \) is the matrix of coefficients defined as

\[
\Phi = \begin{bmatrix} \phi^V \\ \phi^H \end{bmatrix}, \quad \phi^V = [\phi^V_{ij}]_{i=1,...,m, \ j=1,...,n}, \quad \phi^H = [\phi^H_{ij}]_{i=1,...,m, \ j=1,...,n}
\]

and \( b \) is the array of the right–hand side

\[
b = \begin{bmatrix} b^V \\ b^H \end{bmatrix}, \quad b^V = [b^V_1, \ldots, b^V_m]^T, \quad b^H = [b^H_1, \ldots, b^H_m]^T.
\]
**Case 2:** We represent \( \sigma(z) \) by using a linear combination of linear B-splines,

\[
\sigma(z) \simeq \sum_{j=0}^{n} \alpha_j \ B_j(z),
\]

where

\[
B_j(z) = \begin{cases} 
\frac{1}{\delta_j} (z - z_{j-1}), & z_{j-1} \leq z < z_j, \\
\frac{1}{\delta_{j+1}} (z_{j+1} - z), & z_j \leq z < z_{j+1}, \\
0, & \text{otherwise},
\end{cases}
\]

with

\[
B_0(z) = \begin{cases} 
\frac{1}{\delta_1} (z_1 - z), & z_0 \leq z < z_1, \\
0, & \text{otherwise},
\end{cases}
\]

and \( \delta_j = z_j - z_{j-1} \) and \( j = 0, 1, \ldots \).
By a similar procedure, we get a matrix system

$$\Phi \alpha = b$$

where

$$\phi_{ij}^V = \begin{cases} 
\frac{1}{\sqrt{4(h_i + z_0)^2 + 1}} - \frac{1}{\delta_1} \psi^V(h_i + z_0, h_i + z_1), & \text{if } j = 0 \\
\frac{1}{\delta_j} \psi^V(h_i + z_{j-1}, h_i + z_j) - \frac{1}{\delta_{j+1}} \psi^V(h_i + z_j, h_i + z_{j+1}), & \text{if } j \geq 1,
\end{cases}$$

with

$$\psi^V(a, b) = \frac{\sinh^{-1}(2b) - \sinh^{-1}(2a)}{2}.$$  

**Remark:** Similar expression holds for the horizontal orientation.
**Case 3:** We introduce the change of variable $y = z/\zeta$, which leads to

$$
\zeta \int_{0}^{1} k^{V}(\zeta y + h)\sigma(\zeta y)\,dy = \tilde{g}^{V}(h).
$$

Now, we approximate the integrand by a Bernstein polynomial

$$
k^{V}(\zeta y + h)\sigma(\zeta y) = \sum_{j=0}^{n} k^{V}(\zeta y_{j} + h)\sigma(\zeta y_{j}) p_{nj}(y),
$$

where $y_{j} = j/n$, $j = 0, 1, \ldots, n$, and

$$
p_{nj} = \binom{n}{j} z^{j} (1 - z)^{n-j}.
$$
By substituting, and collocating the resulting equation at the heights $h_i$, $i = 1, \ldots, m$, we obtain

$$\sum_{j=0}^{n} k^V(\zeta y_j + h_i) d_j \sigma(\zeta y_j) = \tilde{g}^V(h_i), \quad i = 1, \ldots, m,$$

where

$$d_j = \zeta \int_0^1 p_{n,j}(y) \, dy = \zeta \binom{n}{j} B(j + 1, n - j + 1) = \frac{\zeta}{n + 1},$$

being $B(x, y)$ the beta function.

Then, by setting

$$\phi_{ij}^V = \frac{\zeta}{n + 1} k^V(\zeta y_j + h_i), \quad \sigma_j = \sigma(\zeta y_j), \quad b_i^V = \tilde{g}^V(h_i),$$

and repeating the same procedure for the second integral equation, we get the linear system

$$\Phi \sigma = b.$$
A way to treat the ill-conditioning of $\Phi$ is to derive a new problem with a well-conditioned rank-deficient coefficient matrix.

The best rank $k$ approximation $\Phi_k$, according to the 2-norm, can be obtained by the SVD decomposition, $\Phi = U\Sigma V^T$. The truncated SVD regularization solves the problem

$$\begin{cases} 
\min_{\sigma} \|\sigma\|, \\
\text{subj. to } \min_{\sigma} \|\Phi\sigma - b\|,
\end{cases}$$

and the minimal norm solution can be written as

$$\sigma_k = \sum_{i=1}^{k} \frac{u_i^T b}{\sigma_i} v_i,$$

where $k$ is the truncation parameter.
Let us introduce a regularization matrix \( L \in \mathbb{R}^{p \times n} \) \((p \leq n)\). Common choices are \( L = D_1, D_2 \).

The minimal \( L \)-norm solution is the vector which solves

\[
\begin{align*}
\min_{\sigma} & \quad \| L \sigma \|, \\
\text{subj. to} & \quad \min_{\sigma} \| \Phi \sigma - b \|,
\end{align*}
\]

under the assumption \( \mathcal{N}(\Phi) \cap \mathcal{N}(L) = \{0\} \).

By the generalized singular value decomposition (GSVD) of \((\Phi, L)\)

\[
\Phi = U \begin{bmatrix} \sum & 0 \\ 0 & I_{n-p} \end{bmatrix} Z^{-1}, \quad L = V \begin{bmatrix} M & 0 \end{bmatrix} Z^{-1},
\]

we compute the truncated GSVD (TGSVD) solution \( \sigma_k \), where \( k \leq p \) is the regularization parameter.
Let us introduce a regularization matrix $L \in \mathbb{R}^{p \times n}$ ($p \leq n$). Common choices are $L = D_1, D_2$.

The minimal $L$-norm solution is the vector which solves

$$
\begin{cases}
\min_{\sigma} \|L\sigma\|,
\quad \\
\text{subj. to } \min_{\sigma} \|\Phi\sigma - b\|,
\end{cases}
$$

under the assumption $\mathcal{N}(\Phi) \cap \mathcal{N}(L) = \{0\}$.

By the generalized singular value decomposition (GSVD) of $(\Phi, L)$

$$
\Phi = U \begin{bmatrix} \sum & 0 \\ 0 & I_{n-p} \end{bmatrix} Z^{-1}, \quad L = V \begin{bmatrix} M & 0 \end{bmatrix} Z^{-1},
$$

we compute the truncated GSVD (TGSVD) solution

$$
\sigma_k = \sum_{i=p-k+1}^{p} \frac{u_i^T b}{\sigma_i} z_i + \sum_{i=p+1}^{n} (u_i^T b) z_i,
$$

The choice of the reg. parameter $k$ is crucial: $k^* = \arg \min_k \|\sigma_k - \sigma\|$.
Effectiveness of regularization

\[ m = 20; \quad n = 30; \quad \eta = 0; \quad \zeta = 6; \quad \sigma^a(z) = e^{-(z-1)^2} + 1 \]

left: no regularization; right: TSVD regularization
Numerical results

Comparison between $L = I$ and $L = D_1$

\[ m = 20; \; n = 500; \; \eta = 0; \; \zeta = 30, \; \sigma^a(z) = e^{-(z-1)^2} + 1 \]

left: $L = I$; right: $L = D_1$
Numerical results

Comparison between two different profiles of the conductivity

\[ m = 20; \ n = 500; \ \eta = 0; \ \zeta = 30 \]

left: \( \sigma^b(z) = \begin{cases} 
0.2, & z \in (\infty, 0.5), \\
2, & z \in [0.5, 1.5], \\
0.2, & z \in (1.5, \infty). 
\end{cases} \)

right: \( \sigma^a(z) = e^{-(z-1)^2} + 1 \)
Numerical results

Different regularization matrix

$m = 20; \ n = 500; \ \eta = 10^{-3}; \ \zeta = 30; \ \sigma^a(z) = e^{-(z-1)^2} + 1$

left: $L = D_1$; right: $L = D_2$
Numerical results

Different noise levels

\[ m = 20; \quad n = 500; \quad L = D_1; \quad \zeta = 30; \quad \sigma^b(z) = \begin{cases} 
0.2, & z \in (-\infty, 0.5), \\
2, & z \in [0.5, 1.5], \\
0.2, & z \in (1.5, \infty). \end{cases} \]

left: \( \eta = 10^{-6} \); right: \( \eta = 10^{-3} \)
THANK YOU FOR YOUR ATTENTION!