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Numerical Linear Algebra Aspects in the Analysis of Absorption Graphs

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MAIN TOPICS

- Motivation for this study
- The mathematical tools
- Numerical methods for the computation of the absorption inverse and related quantities
- Centrality measures for graphs with absorption
- Numerical experiments

The notion of Absorption Graph

$(\mathbf{G}, \mathbf{d})$, $\mathbf{d} = [\mathbf{d_1}, \dots, \mathbf{d_n}]$

We suppose that

- each node of the graph represents a transient state in a Markov process,
- each transient state comes with a transition rate $d_i > 0$ to an absorbing state (labeled $n + 1$).

$\hookrightarrow$ arise in epidemiology, ecology,
in the modelling of disease spreading in community networks, etc.

The notion of the Absorption Graphs

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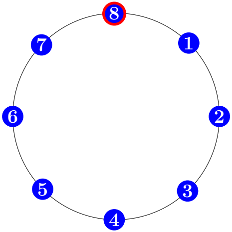

Figure: C_8 with a very large absorption rate at node 8.

-The distance between 1 and 7 has effectively increased
-The graph has essentially changed from a circle to a path

↪ **metrics that take account both graph structure and absorptions (absorption inverse)**

The absorption inverse

Let $\mathbf{L} \in \mathbb{R}^{\mathbf{n} \times \mathbf{n}}$ be the outdegree Laplacian matrix ($\mathbf{L} = \mathbf{W} - \mathbf{A}$) associated with $\mathbf{G}$.

The **absorption inverse $\mathbf{L}^{\mathbf{d}}$** introduced in
K. A. Jacobsen, J. H. Tien, A generalized inverse for graphs with absorption, Linear Algebra Appl., 537 (2018), pp. 118–147.

is a generalized inverse of $\mathbf{L}$ that

- captures valuable information about the graph, combining
  - the known node absorption rates $\mathbf{d_i}$
  - the structural properties of the graph $\mathbf{G}$ encoded in $\mathbf{L}$.
- can be used (especially $\mathbf{L}^{\mathbf{d}}\mathbf{b}$, $\mathbf{b} \in \mathbb{R}^{\mathbf{n}}$)
  - to define a notion of distance,
  - for graph partitioning purposes,
  - as a centrality measure for ranking the nodes.

The absorption inverse

The *absorption inverse of* $\mathbf{L}$ *with respect to* $\mathbf{d}$, denoted as $\mathbf{L^d}$, is a $\{1,2\}$-inverse of $\mathbf{L}$ which satisfies the following conditions:

$$\begin{aligned} \mathbf{L^d L y} &= \mathbf{y}, \quad \text{for } \mathbf{y} \in \mathbf{N_{1,0}}, \\ \mathbf{L^d y} &= \mathbf{0}, \quad \text{for } \mathbf{y} \in \mathbf{R_{1,0}}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{N_{1,0}} &= \{\mathbf{x} \in \mathbb{R}^{\mathbf{n}} : \mathbf{Dx} \in \mathrm{Range}(\mathbf{L})\}, \\ \mathbf{R_{1,0}} &= \{\mathbf{Dx} : \mathbf{x} \in \mathrm{Ker}(\mathbf{L})\}, \end{aligned}$$

and $\mathbf{D}$ is the diagonal matrix whose entries are the absorption rates $\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_n}$.

↪ For a strongly connected graph, $\mathbf{L^d}$ **exists and is unique**.

Properties of the absorption inverse

* $\mathbf{L^d} = \mathbf{D^{-1}}(\mathbf{LD^{-1}})^{\#}$ where $^{\#}$ denotes the group inverse.

* $\mathbf{L^d}$ is unaffected under scaling of the absorption rates.

↪ The case $(\mathbf{G}, \mathbf{1})$ is equivalent to a “standard” graph and $\mathbf{L^d} = \mathbf{L^{\#}}$

* The matrices $\mathbf{LL^d}$ and $\mathbf{L^dL}$ are similar under $\mathbf{D}$,

$$\mathbf{DL^dLD^{-1}} = \mathbf{LL^d}.$$

* $\mathbf{L^d} = (\mathbf{I_n} - \mathbf{VD})\mathbf{Y}(\mathbf{I_n} - \mathbf{DV})$

where $\mathbf{Y}$ is any $\{1\}$-inverse of the Laplacian matrix $\mathbf{L}$, $\mathbf{V} = \mathbf{v1^T}/\tilde{\mathbf{d}}$, $\mathbf{v} \in \mathrm{Ker}(\mathbf{L})$ with $\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{v_i} = \mathbf{1}$, and $\tilde{\mathbf{d}} = \mathbf{d^T v}$.

↪ $\mathbf{L^d}$, being unique, does not depend on the choice of $\mathbf{Y}$

Mathematical Tools

- $\mathbf{L^d = (I_n - VD)Y(I_n - DV)}$

- The factorization $\mathbf{L = \tilde{L}HU}$,

where $\mathbf{\tilde{L}}$ and $\mathbf{U}$ are unit lower and upper triangular matrices, respectively, and $\mathbf{H} = \mathrm{diag}(\mathbf{h_1, h_2, \ldots, h_{n-1}, 0})$, $\mathbf{h_i > 0}$.

↪ The matrix

$$\mathbf{L^- = U^{-1}H^-\tilde{L}^{-1}}$$

where $\mathbf{H^-} = \mathrm{diag}(\mathbf{h_1^{-1}, h_2^{-1}, \ldots, h_{n-1}^{-1}, 0})$,

is a $\{1,2\}$ - **generalized inverse of** $\mathbf{L}$.

$$\hookrightarrow \quad \mathbf{L^d = (I_n - VD)L^-(I_n - DV)}$$

Direct Algorithm for L^d

Michele Benzi*, Paraskevi Fika+, Marilena Mitrouli+

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Algorithm 1: Direct Algorithm for L^d

Input: $L \in \mathbb{R}^{n\times n}$ a Laplacian matrix, $d \in \mathbb{R}^n$ a vector of absorption rates

Output: L^d

-Obtain the triangular factorization $L = \tilde{L}HU$

-Find the normalized vector $v \in \mathrm{Ker}(L)$

-Compute the matrices

$$L^- = U^{-1}H^-\tilde{L}^{-1}$$

$$Y_1 = VDL^-, \quad Y_2 = L^-DV \quad \text{and} \quad Y_3 = Y_1DV$$

Return $L^d = L^- - Y_1 - Y_2 + Y_3$

- For undirected graphs: $\tilde{L} = U^T$ and $v = (1/n)\mathbf{1} \in \mathrm{Ker}(L)$.

Direct Algorithm for $L^d b$

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Algorithm 2: Direct method for computing $L^d b$

Input: $L \in \mathbb{R}^{n \times n}$ a Laplacian matrix, $d \in \mathbb{R}^n$ a vector of absorption rates, $b \in \mathbb{R}^n$ a vector

Output: $L^d b$

-Obtain the triangular factorization $L = \tilde{L} H U$

-Find the normalized vector $v \in \mathrm{Ker}(L)$

-Compute the vectors

$c = b - DVb$ and

$x = L^{-} c$

Return $L^d b = x - v^T (d^T x)/\tilde{d}$

- The matrices D, V, L^{-} are not formed explicitly.

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Iterative Algorithms for $L^d b$

$$L^d b = (I_n - VD)L^+(I_n - DV)b$$

The system $Lx = c$, for $c = b - DVb$, is consistent.

Undirected case

Algorithm 3: Iterative method for computing $L^d b$

Input: $L \in \mathbb{R}^{n\times n}$ a Laplacian matrix, $d \in \mathbb{R}^n$ a vector of absorption rates, $b \in \mathbb{R}^n$ a vector

Output: $L^d b$

-Compute the normalized vector $v = (1/n)\mathbf{1}$

-Compute the vector $c = b - DVb$

-Solve the system $Lx = c$ using the preconditioned conjugate gradients method

Return $L^d b = x - v^T(d^T x)/\tilde{d}$

Iterative Algorithms for $L^d b$

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Algorithm 4: Iterative methods for computing $L^d b$

Input: $L \in \mathbb{R}^{n\times n}$ a Laplacian matrix, $d \in \mathbb{R}^n$ a vector of absorption rates, $b \in \mathbb{R}^n$ a vector

Output: $L^d b$

-Solve the system $Lv = \mathbf{0}$ using the preconditioned gmres

-Compute the vector $c = b - DVb$

-Solve the system $Lx = c$ using the preconditioned gmres

Return $L^d b = x - v^T(d^T x)/\tilde{d}$

Updating/downdating techniques

Let us suppose that we have an updating in the graph $A_1 = A + e_i e_j^T$. Then $L_1 = L - (e_i - e_j)e_j^T$.

The generalized inverse of L_1, is

$$L_1^- = L^- - P,$$

$P = (q/q_j)L^-_{j:}$, $L^-_{j:}$ denotes the jth row of L^- and $q = v/v_n$.
In the case of **balanced graphs**, $P = \mathbf{1}L^-_{j:}$

The absorption inverse of L_1, is

$$L_1^d = L^d - (I_n - VD)L^- DK - KD(I_n - M)L^-(I_n - D(V-K)),$$

$$K = \frac{Vk_1 - K_2}{d^T v - k_1},\ k_1 = d^T N,\ K_2 = N\mathbf{1}^T,\ N = \frac{v_j(L^-_{:i} - L^-_{:j})}{L^-_{j,i} - L^-_{j,j} - 1}$$

$M = \frac{1}{q_j} q e_j^T$ and $L^-_{:j}$ denotes the jth column of the matrix L^-.

Graph centrality measures

- $L^d\mathbf{1}$ (Jacobsen & Tien, LAA 2018)

- gives relatively little weight to the graph topology,
(the absorption rates "weigh" more for the node ranking)

- when the absorption rates are all equal it holds $L^d\mathbf{1} = \mathbf{0}$

- does not reduce to any known centrality measure for "standard graphs"

- We consider $L^dW\mathbf{1}$=L^dw and $\mathrm{diag}(L^dW)$
$W = \mathrm{diag}(w)$, w_i is the sum of outgoing weights of node i

- give more weight to the topology of the graph

- in the limit of equal absorption rates the centrality measures are still able to discriminate between nodes

Graph centrality measures

Balanced graph: the indegrees of each node equal the outdegrees

Unbalanced graphs

- $L^d W_s \mathbf{1}$, for general centrality ranking

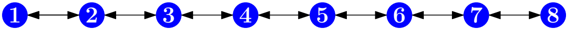
  where $W_s = W_o + W_i$,

  W_o is the diagonal matrix with the weighted outdegrees

  W_i is the diagonal matrix with the weighted indegrees

- $L^d W_o \mathbf{1}$, for ranking hubs (broadcaster)
- $L^d W_i \mathbf{1}$, for ranking authorities (receiver)

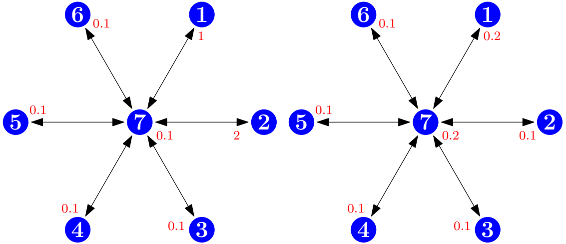
Graph centrality measures



Node	Case 1	Case 2	
	$\mathbf{L^dW1}$	$\mathbf{L^dW1}$	$\mathbf{L^d1}$
1	-0.88	-1.63	-0.41
2	-0.13	-1.07	-0.52
3	0.38	-0.96	-0.74
4	**0.63**	0.26	-0.19
5	**0.63**	1.04	0.26
6	0.38	**1.37**	0.59
7	-0.13	1.26	0.81
8	-0.88	0.70	**0.93**

- Case 1: $d = \mathbf{1}$
- Case 2: $d = [1, 1, 2, 1, \ldots, 1]^T$

Graph centrality measures



Node	Case 1	Case 2		Case 3	
	$\mathbf{L^d W 1}$	$\mathrm{diag}(\mathbf{L^d W})$	$\mathbf{L^d 1}$	$\mathbf{L^d W 1}$	$\mathbf{L^d 1}$
1	-0.10	0.84	0.91	-1.11	-0.56
2	-0.10	0.27	-1.09	0.22	**0.22**
3	-0.10	1.35	**2.71**	0.22	**0.22**
4	-0.10	1.35	**2.71**	0.22	**0.22**
5	-0.10	1.35	**2.71**	0.22	**0.22**
6	-0.10	1.35	**2.71**	0.22	**0.22**
7	**0.61**	**2.47**	1.91	**0.56**	0

Undirected case

Network	n	edges	Jacobi		SGS	
			its	Time	its	Time
cs4	22499	87716	220	1.64e-1	86	1.29e-1
22July06	22963	96872	62	5.68e-2	30	5.28e-2
fetooth	78136	905182	242	9.78e-1	108	9.91e-1
ferotor	99617	1324862	278	1.49e0	123	1.54e0
feocean	143437	819186	599	4.21e0	212	2.56e0
coAuthorsCiteseer	227320	1628268	305	4.10e0	124	3.48e0
citationCiteseer	268495	2313294	243	4.22e0	101	3.66e0
coAuthorsDBLP	299067	1955352	173	3.16e0	70	2.63e0
auto	448695	6629222	433	1.35e1	191	1.28e1
coPapersDBLP	540486	30491458	224	1.91e1	93	1.84e1
tx2010	914231	4456272	2616	1.30e2	1061	9.95e1
NACA0015	1039183	6229636	3712	2.02e2	1529	1.56e2
belgiumosm	1441295	3099940	14722	9.20e2	5462	5.33e2
netherlandsosm	2216688	4882476	25219	2.57e3	9521	1.46e3
M6	3501776	21003872	6261	1.12e3	2478	7.96e2
333SP	3712815	22217266	8876	1.66e3	3599	1.20e3
venturiLevel3	4026819	16108474	10079	1.80e3	3623	1.02e3

Undirected case

Network	ILU		block - LU-8		block - LU-16	
	its	Time	its	Time	its	Time
cs4	77	1.26e-1	108	1.59e-1	125	3.44e-1
22July06	22	4.14e-2	58	7.93e-2	59	7.22e-2
fetooth	84	8.05e-1	162	1.46e0	190	1.59e0
ferotor	101	1.28e0	162	1.95e0	186	2.12e0
feocean	206	2.47e0	253	3.03e0	280	3.30e0
coAuthorsCiteseer	48	1.41e0	106	2.77e0	109	2.79e0
citationCiteseer	79	2.95e0	104	3.30e0	117	3.46e0
coAuthorsDBLP	34	1.40e0	88	2.97e0	92	3.01e0
auto	154	1.05e1	219	1.46e1	256	1.69e1
coPapersDBLP	40	1.07e1	74	1.34e1	100	1.69e1
tx2010	784	7.34e1	925	8.48e1	982	9.10e1
NACA0015	1132	1.10e2	1279	1.23e2	1293	1.26e2
belgiumosm	2613	2.57e2	2886	2.76e2	3120	3.03e2
netherlandsosm	5108	7.83e2	5442	8.23e2	5692	8.77e2
M6	1827	6.13e2	2114	6.77e2	2119	6.90e2
333SP	2653	9.05e2	3228	1.07e3	3501	1.18e3
venturiLevel3	3282	9.62e2	3579	1.04e3	3564	1.02e3

Undirected case

Network	n	nodes of LCC	edges of LCC	Jacobi	
				its	Time
roadNet-CA	1971281	1957027	5520776	8342	7.76e2
roadNet-PA	1090920	1087562	3083028	5665	2.98e2
roadNet-TX	1393383	1351137	3758402	10301	6.60e2
as-Skitter	1696415	1694616	22188418	357	5.57e1
hollywood-2009	1139905	1069126	113682432	157	6.42e1
packing-500x100x100-b050	2145852	2145839	34976486	1062	1.54e2

Network	SGS		ILU		block - LU-8	
	its	Time	its	Time	its	Time
roadNet-CA	3266	5.20e2	2644	4.37e2	2795	4.49e2
roadNet-PA	2098	1.89e2	1793	1.69e2	1931	1.73e2
roadNet-TX	4088	4.49e2	3272	3.63e2	3457	3.83e2
as-Skitter	167	5.09e1	163	5.27e1	265	6.18e1
hollywood-2009	68	7.07e1	18	1.01e2	42	6.67e1
packing-500x100x100-b050	388	1.17e2	340	1.03e2	382	1.15e2

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ca-HepPh	12008	11204	235268	[57, 86]	3.85e-1
email-Enron	36692	33696	361622	[46, 83]	7.31e-1
p2p-Gnutella31	62586	14149	50916	[32, 32]	1.11e-1
enron	69244	8271	147353	[45, 43]	1.39e-1
soc-Epinions1	75888	32223	443506	[39, 80]	6.57e-1
soc-Slashdot0811	77360	70355	888662	[34, 42]	6.91e-1
Wordnet3	82670	13755	37497	[98, 132]	6.70e-1
Internet	124651	6437	18327	[444, 473]	1.68e0
Stanford	281903	150532	1576314	[202, 518]	4.17e1
Linux-call-graph	324085	2760	6425	[51, 50]	5.47e-2
cnr-2000	325557	112023	1646332	[401, 1281]	3.99e1
NotreDame-www	325729	34643	179725	[303, 679]	6.63e0
web-NotreDame	325729	53968	304685	[401, 963]	1.33e1
Stanford-Berkeley	683446	333752	4509784	[141, 248]	5.72e1
web-BerkStan	685230	334857	4523232	[540, 1405]	2.81e2
flickr	820878	527476	9357071	[164, 241]	9.42e1
eu-2005	862664	752725	17933415	[224, 963]	4.17e2
wikipedia-20051105	1634989	1103453	18245140	[39, 41]	2.77e1
wiki-Talk	2394385	111881	1477893	[26, 28]	7.83e-1
wikipedia-20061104	3148440	2104115	36125805	[39, 36]	5.29e1

Directed case

Network	SGS		ILU	
	its	Time	its	Time
ca-HepPh	[32, 56]	2.66e-1	[12, 15]	1.50e-1
email-Enron	[34, 52]	5.73e-1	[14, 19]	2.17e-1
p2p-Gnutella31	[17, 17]	7.19e-2	[12, 12]	4.77e-2
enron	[28, 28]	1.14e-1	[12, 12]	5.15e-2
soc-Epinions1	[24, 49]	4.91e-1	[13, 22]	2.43e-1
soc-Slashdot0811	[22, 26]	6.08e-1	[11, 15]	3.70e-1
Wordnet3	[54, 68]	3.56e-1	[17, 18]	6.45e-2
Internet	[201, 198]	8.87e-1	[100, 101]	4.94e-1
Stanford	[201, 308]	3.20e1	[89, 561]	4.14e1
Linux-call-graph	[26, 25]	3.35e-2	[17, 17]	1.56e-2
cnr-2000	[202, 1237]	5.29e1	[101, 502]	1.76e1
NotreDame-www	[141, 334]	4.40e0	[88, 255]	2.56e0
web-NotreDame	[195, 512]	8.47e0	[91, 2303]	2.69e1
Stanford-Berkeley	[101, 691]	1.38e2	[101, 391]	7.96e1
Web-BerkStan	[197, 428]	1.03e2	[127, 415]	8.63e1
flickr	[82, 117]	5.41e1	[35, 58]	2.18e1
eu-2005	[106, 715]	3.54e2	[81, 249]	1.31e2
wikipedia-20051105	[23, 25]	2.25e1	[12, 12]	1.07e1
wiki-Talk	[15, 17]	7.81e-1	[10, 10]	6.06e-1
wikipedia-20061104	[22, 22]	4.27e1	[14, 12]	2.63e1

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↪ Easy applicable algorithms for the computation of the absorption inverse

↪ Centrality measures for absorption graphs that consider both the graph structure and the node absorption

Further work

- Study of more centrality measures for node ranking.
- Further properties and applications of absorption graphs.
- Estimation of quantities associated with absorption graphs.

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