The Extended-Rational Arnoldi Method

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July 5th, 2018

NASCA 2018

Plan

1. Motivation
2. The Extended-Rational Krylov subspace

Dynamical system

Linear Time-invariant system (LTI)

$$\begin{cases} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector and $u(t),\ y(t) \in \mathbb{R}^s$ are the input and output vectors respectively of the (LTI) system (1). The matrices $B,\ C^T \in \mathbb{R}^{n\times s}$ and $A \in \mathbb{R}^{n\times n}$.

Dynamical system

In many applications, n has a high value $s \ll n$.

$\Longrightarrow$ It will be then reasonable to look for a reduced model having the form :

Reduced Model

$$\left\{ \begin{array}{rcl} \dot{x}_m(t) & = & A_m\, x_m(t) + B_m\, u(t) \\ y_m(t) & = & C_m\, x_m(t), \end{array} \right. \tag{2}$$

with $A_m \in \mathbb{R}^{m\times m}$, $B_m,\ C_m^T \in \mathbb{R}^{m\times s}$, $x_m(t) \in \mathbb{R}^m$ and $y_m(t) \in \mathbb{R}^s$, with $m \ll n$ and such as :

1. $\|y - y_m\|$ is small.
2. The reduced model keeps the main properties of the original system (1).
3. The computational algorithm shoud be stable, and efficient.

Transfert Function

A classical way of relating the input to output by applying the Laplace transform

$$\mathcal{L}(f)(s) := \int_0^\infty e^{-st} f(t)dt,$$

in the system (1)

$$\begin{cases} s\,X(s) &= A\,X(s) + B\,U(s) \\ Y(s) &= C\,X(s) \end{cases},$$

where $X(s)$, $Y(s)$ and $U(s)$ are the Laplace transforms of $x(t)$, $y(t)$ and $u(t)$, respectively. Eliminating $X(s)$ in the previous two equations, we get

$$Y(s) = H(s)\,U(s),$$

where

$$H(s) = C\,(s\,I_n - A)^{-1}\,B. \qquad (3)$$

The transfer function of the system (1).

- Approach the system (1) is the same as apparoche the function (3).

The Krylov subspaces

Among the most efficient technique, there are the projection methods. Those methods are based on Krylov subspaces.

- The standard Krylov subspace :

$$\mathcal{K}_m(A,B) = Vect\{B, Ab, \ldots, A^{m-1}B\}$$

- The Extended Krylov subspace :

$$\mathbb{K}_m(A,B) = \mathrm{Range}\{A^{-m}B, \ldots, A^{-1}B, B, A\,B, \ldots, A^{m-1}B\}.$$

- The Rational Krylov subspace :

$$\mathbf{K}_m(A,B) = \mathrm{Range}\{B, (A - s_2 I)^{-1}B, \ldots, \prod_{i=2}^{m}(A - s_i I)^{-1}B\}$$

Plan

Extended-Rational Krylov subspace

New approach It consists on a new version more general than the Extended and Rational Krylov subspaces

for $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times s}$,

The extended rational Krylov subspace

$$\mathbb{K}_m^{er}(A,B) = \mathrm{Range}\{\prod_{i=1}^{m}(A-s_i I_n)^{-1} B, \ldots, (A-s_1 I_n)^{-1} B, B, \ldots, A^{m-1} B\}.$$

where $s_2, \ldots, s_m$ are a chosen complex numbers

Algorithm 1 The Extended-Rational Arnoldi algorithm

- Inputs : $A \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times s}$, $\{s_1, \ldots, s_m\} \subset \mathbb{C}$ and m.
- Compute $[V_1, \Lambda] = QR([V, (A - s_1 I)^{-1} V])$, $\mathbb{V}_1 = [V_1]$.
- For $j = 1, \ldots, m$
  1. Set $V_j^{(1)}$: first s columns of V_j ; $V_j^{(2)}$: second s columns of V_j.
  2. $\widetilde{V}_{j+1} = [A\, V_j^{(1)}, (A - s_j I)^{-1}\, V_j^{(2)}]$.
  3. Orthogonalize $\widetilde{V}_{j+1}$ with respect to to $\mathbb{V}_1, \ldots, \mathbb{V}_j$ to get V_{j+1}, i.e.,

     for $i = 1, 2, \ldots, j$

     $H_{i,j} = (V_i)^T\, \widetilde{V}_{j+1}$;

     $\widetilde{V}_{j+1} = \widetilde{V}_{j+1} - V_i\, H_{i,j}$;

     end for
  4. $[V_{j+1},\ H_{j+1,j}] = QR(\widetilde{V}_{j+1})$.
  5. $\mathbb{V}_{j+1} = [\mathbb{V}_j,\ V_{j+1}]$.

  End For.

The Extended-Rational Arnoldi Algorithm

$\mathbb{V}_m = [V_1, \ldots, V_m] \in \mathbb{R}^{n\times 2ms}$ with columns constituing an orthonormal basis of the Extended-Rational block Krylov subspace $\mathbb{K}_m^{er}(A, B)$.

Upper block Hessenberg matrix $\overline{\mathbb{H}}_m$ whose non zero blocks are the $H_{i,j}$.

we also define :

- $\mathbb{H}_m$ by removing the last blok of $\overline{\mathbb{H}}_m$
- $\mathbb{T}_m := \mathbb{V}_m^T A \mathcal{V}_m$

The Extended-Rational Arnoldi Algorithm

The matrix $\mathbb{T}_m$ is of great importance for the model reduction by the projection methods. When we manipulate big size model, the direct calculus of the matrix $\mathbb{T}_m$ is of an elevated cost.

- lets $[B,\ (s_1 I - A)^{-1}B] = V_1\Lambda$ be the QR decomposition of $[B,\ (s_1 I - A)^{-1}B]$

$$[B,(s_1 I - A)^{-1}B] = V_1\Lambda = [V_1^{(1)},\ V_1^{(2)}]\begin{bmatrix}\Lambda_{1,1} & \Lambda_{1,2}\\ 0 & \Lambda_{2,2}\end{bmatrix}.$$

- for $k = 1,\dots,m$, we assume that $H_{k+1,k}$could be written as

$$H_{k+1,k} = \begin{bmatrix}H_{k+1,k}^{(1,1)} & H_{k+1,k}^{(1,2)}\\ 0 & H_{k+1,k}^{(2,2)}\end{bmatrix}.$$

Proposition

As defined previously $\overline{\mathbb{T}}_m$ and $\overline{\mathbb{H}}_m$ we could than :
for $k = 1, \ldots, m$

$$\overline{\mathbb{T}}_m \, \widetilde{e}_{2k-1} = \begin{bmatrix} \overline{\mathbb{H}}_k \\ 0_{2(m-k)s \times 2ks} \end{bmatrix} = \overline{\mathbb{H}}_m \, \widetilde{e}_{2k-1},$$

$$\overline{\mathbb{T}}_m \, \widetilde{e}_2 = \left(s_1 \begin{bmatrix} I_{2s} \\ 0_{2ms \times 2s} \end{bmatrix} \begin{bmatrix} \Lambda_{1,2} \\ \Lambda_{2,2} \end{bmatrix} - \overline{\mathbb{T}}_m \, \widetilde{e}_1 \, \Lambda_{1,2} - \widetilde{e}_1 \, \Lambda_{1,1} \right) (\Lambda_{2,2})^{-1},$$

and

$$\mathbb{T}_{m+1} \, \widetilde{e}_{2k+2} = \left(s_k \overline{\mathbb{H}}_m \, \widetilde{e}_{2k} \quad - \quad \begin{bmatrix} \overline{\mathbb{T}}_k \\ 0_{2(m-k)s \times 2ks} \end{bmatrix} \mathbb{H}_k \, \widetilde{e}_{2k} \right.$$

$$\left. - \quad \mathbb{T}_{m+1} \, \widetilde{e}_{2k+1} \, H^{(1,2)}_{k+1,k} - \widetilde{e}_{2k} \right) (H^{(2,2)}_{k+1,k})^{-1},$$

where $\widetilde{e}_i = e_i \otimes I_{2s}$.

Arnoldi relations types

Useful relationships are satisfied by the Arnoldi procedure. Those equations have an important role for reduced model and for calculate the error norm.

Proposition

Soit $\overline{\mathbb{T}}_m = \mathbb{V}_{m+1}^T \, A \, \mathbb{V}_m$ et $\mathbb{T}_m = \mathbb{V}_m^T \, A \, \mathbb{V}_m$.Then we will have the fellowing relation

$$\begin{aligned} A \, \mathbb{V}_m &= \mathbb{V}_{m+1} \, \overline{\mathbb{T}}_m \\ &= \mathbb{V}_m \, \mathbb{T}_m + V_{m+1} \, T_{m+1,m} \, \mathbb{E}_m^T, \end{aligned}$$

where $T_{m+1,m} = V_{m+1}^{\top} A \mathbb{V}_m$ et $\mathbb{E}_m = [0_{2s\times 2(m-1)s}, I_{2s}]^{\top}$.

Numerical Tests

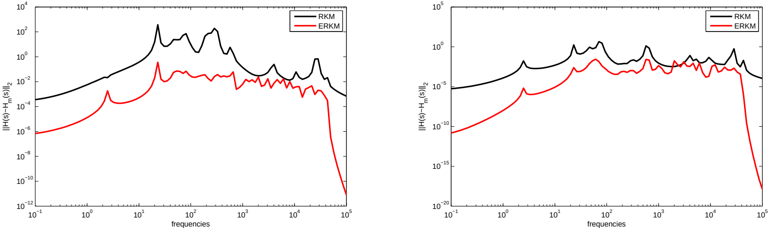

FIGURE – The CDplayer model. The norms of the errors $\|H(i\omega) - H_m(i\omega)\|_2$ for the Rational Block Arnoldi (black) and Extended-Rational Block Arnoldi (Red) for $\omega \in [10^{-1}, 10^5]$ and $s = 2$ with $m = 10$ (left) and $m = 20$ (right).

Numerical Tests

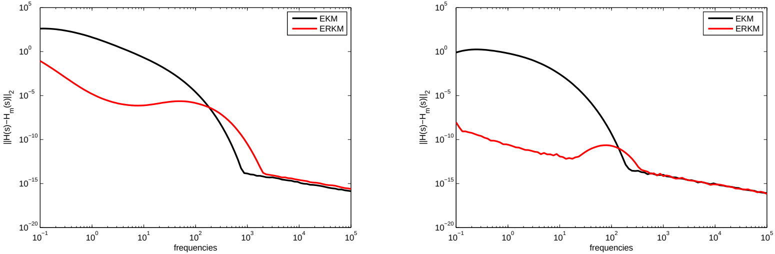

FIGURE – The `Flow` model (n=9669). The norms of the errors $\|H(i\omega) - H_m(i\omega)\|_2$ for the Extended Block Arnoldi (black) and Extended-Rational Block Arnoldi (Red) for the $\omega \in [10^{-1}, 10^5]$ and $s = 3$ with $m = 15$ (left) and $m = 30$ (right).

Thank you for your attention