Envelope: Localization for the Spectrum of a Matrix

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[Based on joint work with Maria Adam, Panos Psarrakos, Katerina Aretaki]

July 2018

Bendixson

Hermitian part of $A \in \mathbb{C}^{n \times n}$:

$$H(A) = \frac{A + A^*}{2}$$

Eigenvalues of $H(A)$:

$$\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$$

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- This infinite intersection of half-planes coincides with the **numerical range** (field of values) of A:

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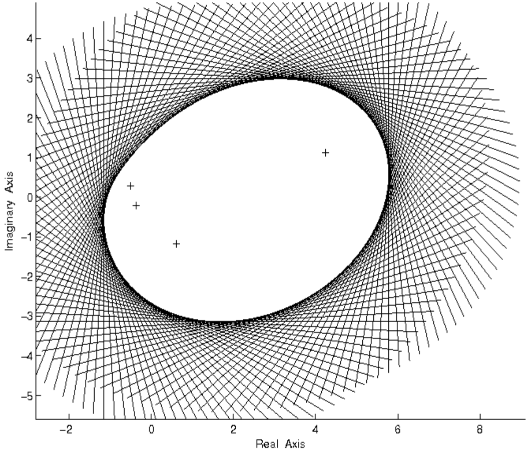
- In fact, this is Johnson's algorithm for computing and plotting the boundary points of $F(A)$.

The numerical range of a 4 × 4 Toeplitz matrix.



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- Replace infinite intersection of half-planes by an infinite intersection of regions in the complex plane (defined by the above cubic curves).
- The outcome is a localization region for the spectrum called the **envelope** of A.
- The envelope is contained in the numerical range and can be quite smaller.

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- $0 \leq \mathrm{v}(A) - \mathrm{u}(A)^2$ is a measure of how close $\delta_1(A) + i\mathrm{u}(A)$ is to being a normal eigenvalue of A.

The inequality

Theorem [Adam and T.]
Every eigenvalue λ of $A \in \mathbb{C}^{n \times n}$ satisfies

$$(\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Im}\lambda - \mathrm{u}(A))^2 \leq$$

$$(\delta_1(A) - \mathrm{Re}\lambda)[\mathrm{v}(A) - \mathrm{u}(A)^2 + (\mathrm{Re}\lambda - \delta_2(A))(\mathrm{Re}\lambda - \delta_1(A))]$$

Proof sketch

(based on A. Sourour deflation - circa 1991)

- $\exists$ unitary U, whose first column is y_1, such that

$$U^* H(A) U = \begin{bmatrix} \delta_1 & 0 \\ 0 & H_1 \end{bmatrix}, \qquad H_1 = \mathrm{diag}\{\delta_2, \delta_3, \ldots, \delta_n\}.$$

- As $U^* S(A) U$ is skew-Hermitian,

$$U^* S(A) U = \begin{bmatrix} \mathrm{i}\,\alpha & u^* \\ -u & S_1 \end{bmatrix},$$

- Consequently, when $\lambda \in \sigma(A)$, the following matrix is singular:

$$U^*(A - \lambda I) U = \begin{bmatrix} \delta_1 + \mathrm{i}\,\alpha - \lambda & u^* \\ -u & H_1 + S_1 - \lambda I \end{bmatrix}.$$

- Schur complement (leading entry) in $U^*(A-\lambda I)U$ is singular:

$$E \;=\; (H_1 + S_1 - \lambda I) + \frac{1}{\delta_1 + \mathrm{i}\,\alpha - \lambda}\; uu^*,$$

- We have

$$H(E) \;=\; H_1 - \mathrm{Re}\lambda I + \frac{\delta_1 - \mathrm{Re}\lambda}{(\delta_1 - \mathrm{Re}\lambda)^2 + (\alpha - \mathrm{Im}\lambda)^2}\; uu^*.$$

- Since $0 \in \sigma(E) \subseteq F(E)$ and $F(H(E)) = \mathrm{Re}F(E)$, it follows $0 \in F(H(E))$, i.e., $\exists$ unit $x \in \mathbb{C}^n$ such that $x^*H(E)x = 0$; i.e.,

$$0 = x^*H_1x - x^*x\,\mathrm{Re}\lambda \;+\; \frac{\delta_1 - \mathrm{Re}\lambda}{(\delta_1 - \mathrm{Re}\lambda)^2 + (\alpha - \mathrm{Im}\lambda)^2}\; x^*uu^*x.$$

- Since $F(H_1) = [\,\delta_n, \delta_2]$ and $F(uu^*) = [0, u^*u]$, we have

$$x^*H_1x \;\le\; \delta_2 \quad \text{and} \quad x^*uu^*x \;\le\; u^*u.$$

- Thus

$$0 \;\le\; \delta_2 - \mathrm{Re}\lambda + \frac{\delta_1 - \mathrm{Re}\lambda}{(\delta_1 - \mathrm{Re}\lambda)^2 + (\alpha - \mathrm{Im}\lambda)^2}\; u^*u,$$

where

$$u^*u = \mathrm{v}(A) - \alpha^2.$$

QED

In and Out Regions

- Theorem gives rise to a cubic algebraic curve $\Gamma(A)$:

$$\left\{ s + \mathrm{i}\, t : \; s, t \in \mathbb{R}, \; \delta_2(A) - s + \frac{(\delta_1(A) - s)(\mathrm{v}(A) - \mathrm{u}(A)^2)}{(\delta_1(A) - s)^2 + (\mathrm{u}(A) - t)^2} = 0 \right\}$$

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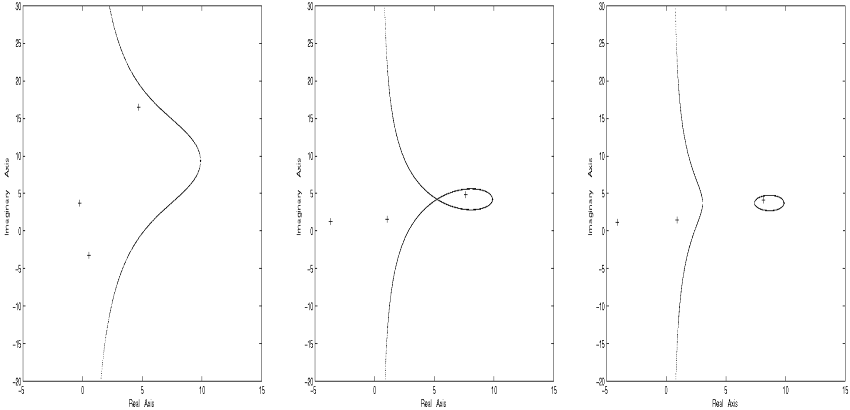
Possible configurations of $\Gamma(A)$



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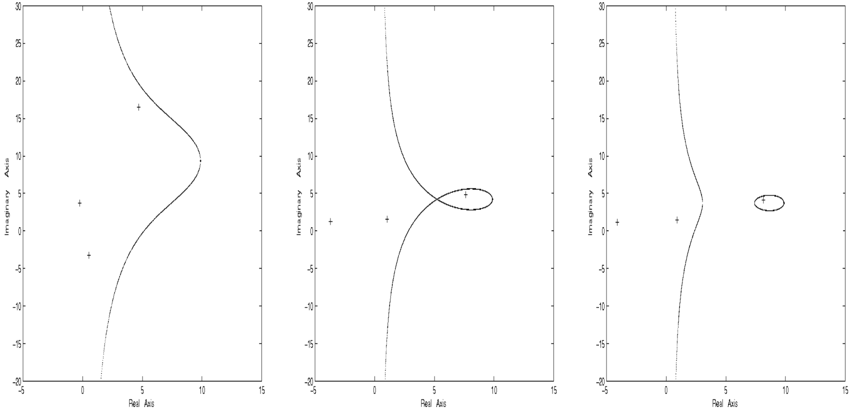
- $\delta_1(A) + \mathrm{i}\,\mathrm{u}(A)$ is a right most point of the numerical range.

See properties of $\Gamma(A)$



The envelope of A

Play the spinning game again: The **envelope** of A is

$$\mathcal{E}(A) = \bigcap_{\theta\in[0,2\pi]} e^{-\mathrm{i}\,\theta}\Gamma_{in}(e^{\mathrm{i}\,\theta}A)$$

Theorem [Psarrakos and T.] For any matrix $A \in \mathbb{C}^{n\times n}$,

$$\sigma(A) \subseteq \mathcal{E}(A) \subseteq F(A)$$

Proof

$$\mathcal{H}_{in}(e^{i\theta}A) = \{e^{-i\theta}(s + i\,t) : s, t \in \mathbb{R} \text{ with } s \leq \delta_1(e^{i\theta}A)\},$$

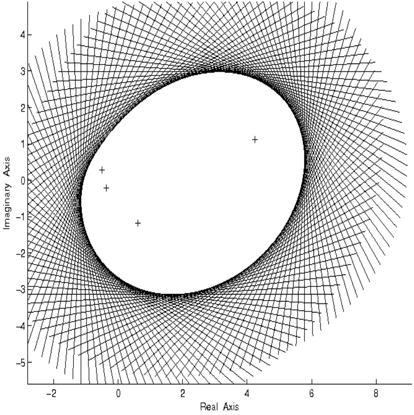
$$F(A) = \bigcap_{\theta \in [0,2\pi]} \mathcal{H}_{in}(e^{i\theta}A)$$

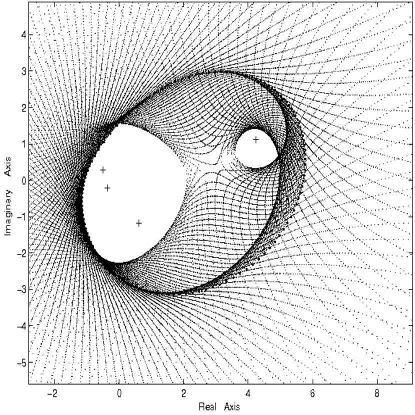
$$\sigma(A) = e^{-i\theta}\sigma(e^{i\theta}A) \subseteq e^{-i\theta}\Gamma_{in}(e^{i\theta}A) \subseteq \mathcal{H}_{in}(e^{i\theta}A)$$

Hence

$$\sigma(A) \subseteq \mathcal{E}(A) = \bigcap_{\theta \in [0,2\pi]} e^{-i\theta}\Gamma_{in}(e^{i\theta}A) \subseteq F(A)$$

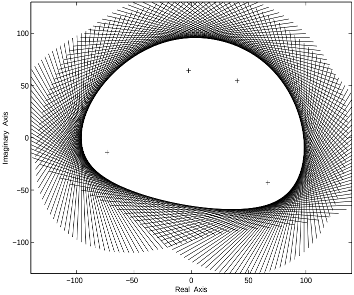
Numerical range and Envelope of a Toeplitz matrix

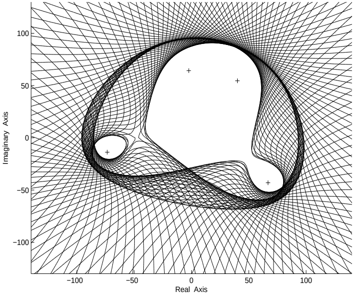




A complex matrix (and a better drawing method)

$$A = \begin{bmatrix} 14 + i\,19 & -4 - i & -55 - i\,13 & -32 + i\,13 \\ 27 + i\,2 & 14 - i\,25 & 64 & 72 \\ 54 + i & 47 - i\,3 & 14 + i\,44 & -32 - i\,42 \\ 76 & 73 & 4 - i\,2 & -11 + i\,24 \end{bmatrix}$$





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- For any $b \in \mathbb{C}$, $\Gamma(A + bI_n) = \Gamma(A) + b$ and $\mathcal{E}(A + bI_n) = \mathcal{E}(A) + b$

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- For every real $r > 0$ and $a \in \mathbb{C}$, $\Gamma(rA) = r\,\Gamma(A)$ and $\mathcal{E}(aA) = a\,\mathcal{E}(A)$

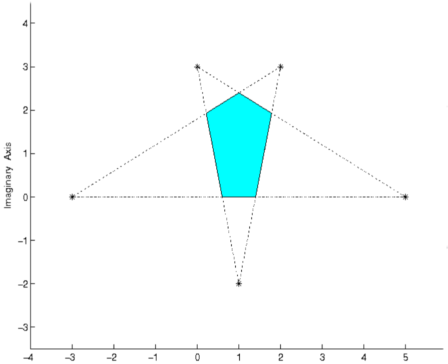
Interesting cases/behavior

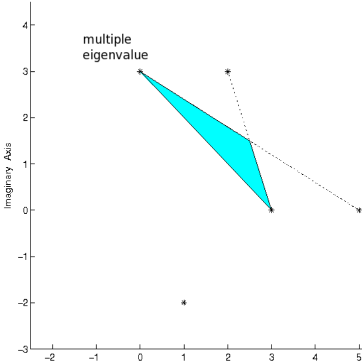
Proposition Let λ_0 be a **simple** eigenvalue of A on the **boundary** of $F(A)$. If λ_0 does **not** lie on a flat portion of $\partial F(A)$, or if it is a non-differentiable point of $\partial F(A)$, then λ_0 is an **isolated point** of the envelope $\mathcal{E}(A)$.

Normal matrices

$$D_1 = \operatorname{diag}\{i\,3, 5, 2 + i\,3, 1 - i\,2, -3\},$$
$$D_2 = \operatorname{diag}\{i\,3, i\,3, 5, 2 + i\,3, 1 - i\,2, 3\}$$

$\mathcal{E}(D_1)$ and $\mathcal{E}(D_2)$ are the shaded regions **union** the isolated points.





Envelope of a normal matrix

- $\lambda_1, \lambda_2, \ldots, \lambda_k$ the simple **extremal** eigenvalues of normal A

(i.e., vertices of $\mathrm{Co}(\sigma(A))$ which must be isolated points of $\mathcal{E}(A)$)

- $\mathcal{C}(A) := \mathcal{E}(A) \setminus \{\lambda_1, \lambda_2, \ldots, \lambda_k\}$

Proposition

(i) If all the eigenvalues of A are simple and extremal, then $\mathcal{E}(A) = \sigma(A)$.

(ii) If all the extremal eigenvalues of A are multiple, then $\mathcal{E}(A) = \mathcal{C}(A) = \mathrm{Co}(\sigma(A)) = F(A)$.

(iii) If $n = 2$ or 3, then $\mathcal{E}(A) = \sigma(A)$.

(iv) Let $n = 4$. If all the eigenvalues of A are extremal, and A does not have two double eigenvalues (for the case of two double eigenvalues, see (ii) above), then $\mathcal{E}(A) = \sigma(A)$.

Envelope of a hermitian matrix

Corollary Let $A \in \mathbb{C}^{n\times n}$ be a hermitian matrix with eigenvalues $\delta_1(A) \geq \delta_2(A) \geq \cdots \geq \delta_n(A)$. Then,

$$\mathcal{E}(A) \;=\; \{\delta_n(A)\}\cup[\delta_{n-1}(A),\delta_2(A)]\cup\{\delta_1(A)\} \;\subseteq\; [\delta_n(A),\delta_1(A)] \;=\; F(A)$$

Envelope of a tridiagonal Toeplitz matrix

$$T_n(c,a,b) = \begin{bmatrix} a & b & \cdots & 0 \\ c & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ 0 & \cdots & c & a \end{bmatrix} \in \mathbb{C}^{n\times n}, \quad bc \neq 0.$$

The numerical range of $T_n(c,a,b)$ coincides with the elliptical disc

$$\{bz + c\overline{z} : \ z \in F(J_n(0))\} + \{a\},$$

where $J_n(0)$ is the $n \times n$ Jordan block with zero eigenvalue.

Theorem The envelope of $T_n(c,a,b) \in \mathbb{C}^{n\times n}$, $bc \neq 0$, is
(1) symmetric with respect to point a.
(2) symmetric with respect to the line

$$\mathcal{L}(T_n(c,a,b)) = \{a + \gamma e^{\mathrm{i}\frac{\arg(b)+\arg(c)}{2}} : \gamma \in \mathbb{R}\}.$$

Envelope of a block-shift matrix

$$A = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_m \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

with $m > 1$ and the zero blocks along the main diagonal being square.

Theorem Let $A \in \mathbb{C}^{n\times n}$ $(n \geq 3)$ be a block-shift matrix. Then $\mathcal{E}(A)$ coincides with the circular disc $\mathcal{D}(0, R)$ centered at the origin, with radius

$$R = \left(\delta_1^2(A) - \left(\sqrt{2\delta_1(A)(\delta_1(A) - \delta_2(A))} - \sqrt{\mathrm{v}(A)}\right)^2\right)^{1/2}.$$

Similarities

Well-known result of Givens for the numerical range:

$$\bigcap \left\{ F(R^{-1}AR) : \ R \in \mathbb{C}^{n \times n}, \det(R) \neq 0 \right\} \ = \ conv\{\sigma(A)\}$$

An analogous result holds for the envelope (long proof if A is not diagonalizable):

$$\bigcap \left\{ \mathcal{E}(R^{-1}AR) : \ R \in \mathbb{C}^{n \times n}, \det(R) \neq 0 \right\} \ \subseteq \ \mathcal{E}(D(A)),$$

where $D(A)$ is the diagonal matrix whose diagonal entries are the eigenvalues of A.

Connection to k-rank Numerical Range

$$\begin{aligned} \Lambda_k(A) &= \left\{\mu \in \mathbb{C} : \; PAP = \mu P \text{ for some rank-}k \text{ orthog. proj. } P \in \mathbb{C}^{n\times n}\right\} \\ &= \left\{\mu \in \mathbb{C} : \; X^*AX = \mu I_k \text{ for some } X \in \mathbb{C}^{n\times k} \text{ such that } X^*X = I_k\right\} \end{aligned}$$

• Connected to the construction of quantum error correction codes for noisy quantum channels...

• Does not necessarily contain all of the eigenvalues of A.

Theorem $\Lambda_{n-1}(A) \subseteq \ldots \subseteq \Lambda_2(A) \subseteq \mathcal{E}(A) \subseteq F(A) = \Lambda_1(A)$

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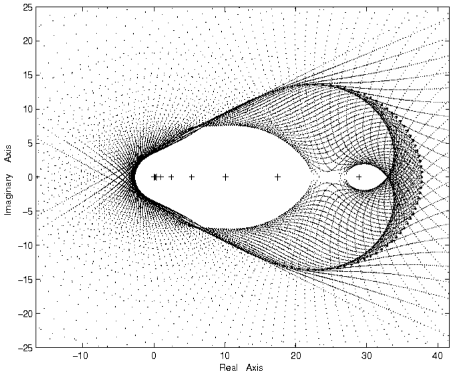
- The improvement expected depends on the **geometry of the eigenvalues**.
- Technique can potentially be generalized to utilize more eigenvalues of $H(A)$.

One last example

The envelope of a Frank matrix (11×11 highly ill-conditioned)



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