On the inverse problem associated to $\mathbf{RA} = \mathbf{A}^{s+1}\mathbf{R}$ when $\mathbf{R}^k = \mathbf{I}$

Leila Lebtahi Óscar Romero Néstor Thome

Facultat de Ciències Matemàtiques, Dpt. de Matemàtiques
Universitat de València, Spain

Dpto de Comunicaciones, Universitat Politècnica de València, Spain

Instituto Universitario de Matemática Multidisciplinar
Universitat Politècnica de València, Spain

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Outline

1. Characterizations of $\{\mathbf{R}, s+1, k\}$-potent matrices
2. Computing $\{\mathbf{R}, s+1, k\}$-potent matrices
3. Algorithm for finding matrices $\mathbf{R}$ such that $\mathbf{A}$ is a $\{\mathbf{R}, s+1, k\}$-potent
4. Numerical example and conclusions

Definition of $\{\mathbf{R}, s+1, k\}$-potent matrices

Definition

Let $s \in \{1, 2, 3, \dots\}$ and $\mathbf{R} \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix, that is, $\mathbf{R}^k = I$. A matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is called $\{\mathbf{R}, s+1, k\}$-potent if it satisfies

$$\mathbf{RA} = \mathbf{A}^{s+1}\mathbf{R}$$

Extends to:

Matrices	**Condition**
idempotent: $\mathbf{A}^2 = \mathbf{A}$	$\mathbf{R} = \mathbf{I}, s = 1$
$\{s+1\}$-potent :$\mathbf{A}^{s+1} = \mathbf{A}$	$\mathbf{R} = \mathbf{I}$
involutory: $\mathbf{A}^2 = \mathbf{I}$	$\mathbf{R} = \mathbf{I}, s = 2$, $\mathbf{A}$ nonsingular
centrosymmetric: $\mathbf{AR} = \mathbf{RA}$	$\mathbf{R} = \mathbf{J}$ is an exchange matrix, $s = 0$
mirrorsymmetric: $\mathbf{AR} = \mathbf{RA}$	$\mathbf{R} = \mathbf{I} \oplus \mathbf{J} \oplus \mathbf{I}$ (on sec. diag.), $s = 0$

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Characterizations of $\{\mathbf{R}, s+1, k\}$-potent matrices:

L-S-T-W, Matrices $\mathbf{A}$ such that $\mathbf{RA} = \mathbf{A}^{s+1}\mathbf{R}$ when $\mathbf{R}^k = \mathbf{I}$,
Linear Algebra and its Applications 439 (2013) 1017–1023

Construction of a group from $\{\mathbf{R}, s+1, k\}$-potent matrices:

C-L-S-T, On a matrix group constructed from an $\{\mathbf{R}, s+1, k\}$-potent matrix,
Linear Algebra and its Applications 461 (2014) 200–210

Direct Problem: How to construct this class of matrices?

Construction of $\{\mathbf{R}, s+1, k\}$-potent matrices for a given $\{k\}$-involutory matrix $\mathbf{R}$.

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Two problems related to $\{\mathbf{R}, s+1, k\}$-potent matrices

Characterizations of $\{\mathbf{R}, s+1, k\}$-potent matrices:

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Construction of $\{\mathbf{R}, s+1, k\}$-potent matrices for a given $\{k\}$-involutory matrix $\mathbf{R}$.

Inverse Problem: How to compute $\{k\}$-involutory matrices $\mathbf{R}$?

Design of an algorithm to construct matrices $\mathbf{R}$ such that a given matrix $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent.

Constructing a bijection

Let

- $S = \{1, 2, \ldots, n_s - 1\}$ where $n_s = (s+1)^k - 1$
- $\varphi : S \cup \{0\} \to S \cup \{0\}$
- $\varphi(j) := b_j$
- b_j is the smallest nonnegative integer such that

$$b_j \equiv j(s+1) \ [\mathrm{mod}\ (n_s)]$$

Then φ is a bijective function.

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Characterizations of $\{\mathbf{R}, s+1, k\}$-potent matrices ...

Theorem

Let

- $\mathbf{A} \in \mathbb{C}^{n \times n}$, $s \in \{1, 2, 3, \dots\}$
- $\mathbf{R} \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix.

Then the following conditions are equivalent:

(a) $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent.

(b) $\boxed{\mathbf{A} \text{ is diagonalizable}}$, $\boxed{\sigma(\mathbf{A}) \subseteq \{0\} \cup \Omega_{n_s}}$, $\boxed{\mathbf{R}\mathbf{P}_0\mathbf{R}^{-1} = \mathbf{P}_0}$,

$\boxed{\mathbf{R}\mathbf{P}_{\varphi(j)}\mathbf{R}^{-1} = \mathbf{P}_j}$ where $j \in S$ and $\boxed{\mathbf{R}\mathbf{P}_{n_s}\mathbf{R}^{-1} = \mathbf{P}_{n_s}}$,

$\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_{n_s}$ are the projectors of the spectral decomposition of $\mathbf{A}$ associated to the eigenvalues $0, \omega_{n_s}^1, \dots, \omega_{n_s}^{n_s-1}, 1$, respectively.

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Then the following conditions are equivalent:

(a) $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent.

(c) $\boxed{\mathbf{A}^{n_s+1} = \mathbf{A}}$,

$\boxed{\mathbf{R}\mathbf{P}_0\mathbf{R}^{-1} = \mathbf{P}_0}$, $\boxed{\mathbf{R}\mathbf{P}_{\varphi(j)}\mathbf{R}^{-1} = \mathbf{P}_j}$ where $j \in S$ and $\boxed{\mathbf{R}\mathbf{P}_{n_s}\mathbf{R}^{-1} = \mathbf{P}_{n_s}}$,

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- $\mathbf{A} \in \mathbb{C}^{n\times n}$, $s \in \{1, 2, 3, \dots\}$
- $\mathbf{R} \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix.

Then the following conditions are equivalent:

(a) $\mathbf{A}$ is $\{\mathbf{K}, s+1\}$-potent.

(d) $\boxed{\mathbf{A}^{\#} = \mathbf{A}^{n_s-1}}$,

$\boxed{\mathbf{R}\mathbf{P}_0\mathbf{R}^{-1} = \mathbf{P}_0}$, $\boxed{\mathbf{R}\mathbf{P}_{\varphi(j)}\mathbf{R}^{-1} = \mathbf{P}_j}$ where $j \in S$ and $\boxed{\mathbf{R}\mathbf{P}_{n_s}\mathbf{R}^{-1} = \mathbf{P}_{n_s}}$,

$\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_{n_s}$ are the projectors of the spectral decomposition of $\mathbf{A}$ associated to the eigenvalues $0, \omega_{n_s}^1, \dots, \omega_{n_s}^{n_s-1}, 1$, respectively.

Theorem

Let

- $\mathbf{A} \in \mathbb{C}^{n\times n}$, $s \in \{1, 2, 3, \dots\}$
- $\mathbf{R} \in \mathbb{C}^{n\times n}$ be a $\{k\}$-involutory matrix.

Then the following conditions are equivalent:

(a) $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent.

(e) there are nonsingular matrices: $\mathbf{P} \in \mathbb{C}^{n\times n}$, $\mathbf{C} \in \mathbb{C}^{r\times r}$ such that

$$\boxed{\mathbf{A} = \mathbf{P}\begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix}\mathbf{P}^{-1}} \qquad \boxed{\mathbf{R} = \mathbf{P}\begin{bmatrix} \mathbf{R_1} & \mathbf{O} \\ \mathbf{O} & \mathbf{R_2} \end{bmatrix}\mathbf{P}^{-1}}$$

where $r = \mathrm{rank}(\mathbf{A})$, $\mathbf{R_1} \in \mathbb{C}^{r\times r}$ and $\mathbf{R_2} \in \mathbb{C}^{(n-r)\times(n-r)}$ are both $\{k\}$-involutory and $\mathbf{C}$ is $\{\mathbf{R_1}, s+1, k\}$-potent.

DIRECT PROBLEM:

Construction of $\{R, s+1, k\}$-potent matrices for a given $\{k\}$-involutory matrix R

Computing $\{\mathbf{R}, s+1, k\}$-potent matrices

A simplification

The cases $\mathbf{R} = \mathbf{I}_n$ only provide the well-known results corresponding to $\mathbf{A}^{s+1} = \mathbf{A}$.
Not interesting !

Computing $\{\mathbf{R}, s+1, k\}$-potent matrices

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Not interesting !

Diagonalization of R

Since $\mathbf{R}$ is $\{k\}$-involutory, there is a nonsingular matrix $\mathbf{T} = \begin{bmatrix} t_1 & \dots & t_n \end{bmatrix}$ such that

$$\mathbf{R} = \mathbf{T} \operatorname{diag}\left(\omega_1 \mathbf{I}_{r_1}, \dots, \omega_{\ell-1} \mathbf{I}_{r_{\ell-1}}, \mathbf{I}_\ell\right) \mathbf{T}^{-1}$$

where $r_1 + \cdots + r_\ell = n$ and $\omega_i \in \Omega_k, i = 1, \dots, \ell$.

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where $r_1 + \cdots + r_\ell = n$ and $\omega_i \in \Omega_k, i = 1, \dots, \ell$.

We can assume $r_1 < r_2 < \cdots < r_\ell$. Otherwise, we compute

- $\omega_i \mathbf{R}$ instead of $\mathbf{R}$, $i = 1, \dots, k$ for k odd
- $-\mathbf{R}$ instead of $\mathbf{R}$, for k even

Diagonalization of $\mathbf{A}$

$$\mathbf{A} = \mathbf{S}\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)\,\mathbf{S}^{-1} \qquad \text{with}$$

$$\lambda_i \in \{0\} \cup \Omega_{n_s}, i = 1, \cdots, n, \quad \mathbf{S} = \begin{bmatrix} s_1 & \dots & s_n \end{bmatrix} \quad \text{and} \quad \mathbf{S}^{-1} = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix}$$

Spectral decomposition of $\mathbf{A}$

Denoting $\mathbf{P}_i = s_i y_i^T$ we have $\boxed{\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i}$.

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Spectral decomposition of $\mathbf{A}$

Denoting $\mathbf{P}_i = s_i y_i^T$ we have $\boxed{\mathbf{A} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i}$.

Main idea:

Construction of the s_i's and y_i's in terms of the t_i's.

Construction of the s_i's and y_i's in terms of the t_i's

Since $\mathbf{RP}_{\varphi(j)} = \mathbf{P}_j$ must hold, we can choose

$$\boxed{s_j = \mathbf{R}s_{\varphi(j)}} \quad \text{and} \quad \boxed{\mathbf{R}^T y_j = y_{\varphi(j)}}$$

for $j \in S$ in order to satisfy the equality $\mathbf{RA} = \mathbf{A}^{s+1}\mathbf{R}$.

Example 1

If $\mathbf{R} = \mathbf{TDT}^{-1}$ is a $\{4\}$-involutory matrix and $\mathbf{D} = \mathrm{diag}(i, -i, 1)$

Construction of s_i''s	Construction of matrix $\mathbf{A}$
$s_1 = s'_{\varphi(j)} = t_1 + t_2 + t_3$ $s_2 = s'_j = it_1 - it_2 + t_3$ $s_3 = t_3$	$\mathbf{A} = \omega^j \mathbf{P}_1 + \omega^{\varphi(j)} \mathbf{P}_2 + \mathbf{P}_3$

where $\omega = \omega_{(s+1)^4-1}$ and $j \in S$.

Example 2

If $\mathbf{R} = \mathbf{TDT}^{-1}$ is a $\{4\}$-involutory matrix and $\mathbf{D} = \text{diag}(-1,-1,i,i)$

Construction of s_i''s	Construction of matrix $\mathbf{A}$
$s_1 = s'_{\varphi(j)} = t_1 + t_3$ $s_2 = s'_j = -t_1 + it_3$ $s_3 = s'_{\varphi(a)} = t_2 + t_4$ $s_4 = s'_a = -t_2 + it_4$	$\mathbf{A} = \omega^j \mathbf{P}_1 + \omega^{\varphi(j)} \mathbf{P}_2 + \omega^a \mathbf{P}_3 + \omega^{\varphi(a)} \mathbf{P}_4$

where $\omega = \omega_{(s+1)^4-1}$, $j \in S$, and moreover $a \in S - \{j, \varphi(j)\}$ such that $\varphi(a) \neq a$.

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Example 3

If $\mathbf{R} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$ is a $\{4\}$-involutory matrix and $\mathbf{D} = \mathrm{diag}(-1, -1, -i, i, 1)$

Construction of s_i''s	Construction of matrix $\mathbf{A}$
$s_1 = s'_{\varphi(j)} = t_1 + t_3 + t_4 + t_5$ $s_2 = s'_j = -t_1 - it_3 + it_4 + t_5$ $s_3 = s'_{\varphi(a)} = t_2$ $s_4 = s'_a = -t_2$ $s_5 = s'_b = t_5$	$\mathbf{A} = \omega^j \mathbf{P}_1 + \omega^{\varphi(j)} \mathbf{P}_2 + \omega^a \mathbf{P}_3 + \omega^{\varphi(a)} \mathbf{P}_4 + \mathbf{P}_5$

where $\omega = \omega_{(s+1)^4-1}$, $j \in S$, and moreover $a \in S - \{j, \varphi(j)\}$ such that $\varphi(a) \neq a$ and $b \in S$ such that $\varphi(b) = b$.

ALGORITHM 1

Inputs: An involutory matrix $\mathbf{R}$.

Outputs: A $\{\mathbf{R}, s+1, 2\}$-potent matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ and the projectors $\mathbf{P}_i$.

- Step 1 Diagonalize $\mathbf{R}$ as $\mathbf{R} = \mathbf{T}\ \mathrm{diag}\left(-\mathbf{I}_r, \mathbf{I}_{n-r}\right)\ \mathbf{T}^{-1}$.
- Step 2 If $r > n - r$, change $\mathbf{R}$ to $-\mathbf{R}$ and rearrange as in Step 1.
- Step 3 For $i = 1, \dots, r$, compute $s_{2i-1} = t_i + t_{i+r}$ and $s_{2i} = -t_i + t_{i+r}$.
- Step 4 For $i = 2r+1, \dots, n$, set $s_i = t_i$.
- Step 5 Solve the linear systems $\mathbf{S}y_i = e_i$, where e_i are the canonical basis vectors of $\mathbb{C}^n$ for $i = 1, \dots, n$.
- Step 6 Compute $\mathbf{P}_i = s_i y_i^T$ for $i = 1, \dots, n$.
- Step 7 For $i = 1, \dots, r$, compute $\mathbf{Q}_i = \omega \mathbf{P}_{2i-1} + \omega^{\varphi(1)}\mathbf{P}_{2i}$.
- Step 8 Compute $\mathbf{A} = \sum_{i=1}^{r} \mathbf{Q}_i + \sum_{i=2r+1}^{n} \mathbf{P}_i$.

End

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Numerical example: an $\{\mathbf{R}, 3, 2\}$-potent matrix

$s = 2,\ k = 2,\ n = 4,\ \mathbf{D} = \mathrm{diag}(-1, -1, 1, 1)$

$$\mathbf{R} = \begin{bmatrix} 1.2989 & -1.2069 & 3.5632 & 3.0460 \\ 1.3793 & -2.7241 & 4.1379 & 4.8276 \\ -0.2759 & 1.3448 & -3.8276 & -3.9655 \\ 0.6437 & -2.1379 & 4.5977 & 5.2529 \end{bmatrix}$$

$\mathbf{A} =$

$$\begin{bmatrix} -0.3820 - 0.0731i & 1.2435 - 0.0853i & -0.9103 + 0.4877i & -0.9834 + 1.1582i \\ 0.3657 + 0.5202i & 0.6035 - 0.4145i & -1.0241 - 0.3251i & -1.0180 + 0.4064i \\ -0.2845 - 0.1300i & 0.4145 - 0.7803i & 0.0894 + 0.7884i & -0.6421 + 0.9591i \\ 0.3657 + 0.5202i & -0.1036 + 0.2926i & -1.0241 - 0.3251i & -0.3109 - 0.3007i \end{bmatrix}$$

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INVERSE PROBLEM:

How to compute $\{k\}$-involutory matrices R such that a given matrix A is $\{\mathbf{R}, s+1, k\}$-potent?

Definition

For a given positive integer s, the square, complex matrix $\mathbf{A}$ is called

a *potential* $\{\mathbf{R}, s+1, k\}$*-potent matrix* if $\mathbf{A}^{n_s+1} = \mathbf{A}$,

or equivalently,

if $\mathbf{A}$ is diagonalizable and $\sigma(\mathbf{A})$ is contained in $\{0\} \cup \Omega_{n_s}$.

Observation

- $\mathbf{R}$ is completely unspecified here.
- It is generally much easier and faster to test that $\mathbf{A}^{n_s+1} = \mathbf{A}$ than is to determine the spectrum of $\mathbf{A}$ and to determine that $\mathbf{A}$ is diagonalizable.

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ALGORITHM 2

Inputs: Integers $s \geq 1, k \geq 2$, and $\mathbf{A} \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Output: A decision on whether A is potentially $\{\mathbf{R}, s+1, k\}$-potent or not.

Step 1 If either $\mathbf{A}^{n_s+1} = \mathbf{A}$ or, $\mathbf{A}$ is diagonalizable and $\sigma(\mathbf{A}) \subseteq \{0\} \cup \Omega_{n_s}$, then "$\mathbf{A}$ is potentially $\{\mathbf{R}, s+1, k\}$-potent". Go to End.

Step 2 "$\mathbf{A}$ is not potentially $\{\mathbf{R}, s+1, k\}$-potent, and there is no $\{k\}$-involutory matrix $\mathbf{R} \in \mathbb{C}^{n\times n}$ such that $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent".

End

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Notations

- The notations $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum of two matrices, respectively.
- For any matrix $\mathbf{X} = [x_{ij}] \in \mathbb{C}^{n \times n}$, let $v(\mathbf{X}) = [v_k] \in \mathbb{C}^{n^2 \times 1}$ be the vector formed by stacking the columns of $\mathbf{X}$ into a single column vector.

  The expression $[v(\mathbf{X})]_{\{(j-1)n+1,\ldots,(j-1)n+n\}}$, for $j = 1, \ldots, n$, denotes the j^{th} column of $\mathbf{X}$.
- **Property**: If $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\mathbf{B} \in \mathbb{C}^{n \times n}$ then

$$\mathrm{Ker}(\mathbf{A}) \cap \mathrm{Ker}(\mathbf{B}) = \mathrm{Ker}\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}\right)$$

We recall that the principal idempotents associated with the eigenvalues $\lambda_1, \ldots, \lambda_\ell$ are given by

$$\mathbf{P}_t = \frac{p_t(\mathbf{A})}{p_t(\lambda_t)} \qquad \text{where} \qquad p_t(\eta) = \prod_{\substack{i=1 \\ i \neq t}}^{\ell} (\eta - \lambda_i)$$

By using the function φ and these projectors, it is possible to consider the matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{P}_{\varphi(0)}^T \oplus -\mathbf{P}_0 \\ \mathbf{P}_{\varphi(1)}^T \oplus -\mathbf{P}_1 \\ \vdots \\ \mathbf{P}_{\varphi(n_s-1)}^T \oplus -\mathbf{P}_{n_s-1} \\ \mathbf{P}_{\varphi(n_s)}^T \oplus -\mathbf{P}_{n_s} \end{bmatrix}.$$

Solving the inverse problem: Idea

We focus our attention on solving the matrix equations (in the unknown $\mathbf{R}$):

$$\boxed{\mathbf{R}\mathbf{P}_{\varphi(j)} = \mathbf{P}_j\mathbf{R}}$$

that is, to find the common solutions to

$$\boxed{\mathbf{R}\mathbf{P}_{\varphi(j)} = \mathbf{P}_j\mathbf{R}}, \text{ for } j \in S \cup \{0\} \quad \text{and} \quad \boxed{\mathbf{R}\mathbf{P}_{n_s} = \mathbf{P}_{n_s}\mathbf{R}}$$

After vectoring, the Kronecker product allows us to write

$$v(\mathbf{R}\mathbf{P}_{\varphi(j)}) = v(\mathbf{P}_j\mathbf{R}) \iff (\mathbf{P}_{\varphi(j)}^T \otimes \mathbf{I}_n)v(\mathbf{R}) = (\mathbf{I}_n \otimes \mathbf{P}_j)v(\mathbf{R}),$$

for $j \in S$, and analogously,

$$v(\mathbf{R}\mathbf{P}_{n_s}) = v(\mathbf{P}_{n_s}\mathbf{R}) \iff (\mathbf{P}_{n_s}^T \otimes \mathbf{I}_n)v(\mathbf{R}) = (\mathbf{I}_n \otimes \mathbf{P}_{n_s})v(\mathbf{R}).$$

Solving the inverse problem: Idea

By the property about kernels:

we have to find (non trivial) solutions $v(\mathbf{R})$ of the null space of

$$\begin{bmatrix} (\mathbf{P}_{\varphi(0)}^T \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_0) \\ (\mathbf{P}_{\varphi(1)}^T \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_1) \\ \vdots \\ (\mathbf{P}_{\varphi(n_s-1)}^T \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_{n_s-1}) \\ (\mathbf{P}_{n_s}^T \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_{n_s}) \end{bmatrix}$$

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Solving the inverse problem: Idea

By the property about kernels:

we have to find (non trivial) solutions $v(\mathbf{R})$ of the null space of

$$\begin{bmatrix} (\mathbf{P}^T_{\varphi(0)} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_0) \\ (\mathbf{P}^T_{\varphi(1)} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_1) \\ \vdots \\ (\mathbf{P}^T_{\varphi(n_s-1)} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_{n_s-1}) \\ (\mathbf{P}^T_{n_s} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_{n_s}) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{P}^T_{\varphi(0)} \oplus -\mathbf{P}_0 \\ \mathbf{P}^T_{\varphi(1)} \oplus -\mathbf{P}_1 \\ \vdots \\ \mathbf{P}^T_{\varphi(n_s-1)} \oplus -\mathbf{P}_{n_s-1} \\ \mathbf{P}^T_{\varphi(n_s)} \oplus -\mathbf{P}_{n_s} \end{bmatrix}}_{\mathbf{M}}$$

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By the property about kernels:

we have to find (non trivial) solutions $v(\mathbf{R})$ of the null space of

$$\begin{bmatrix} (\mathbf{P}^T_{\varphi(0)} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_0) \\ (\mathbf{P}^T_{\varphi(1)} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_1) \\ \vdots \\ (\mathbf{P}^T_{\varphi(n_s-1)} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_{n_s-1}) \\ (\mathbf{P}^T_{n_s} \otimes \mathbf{I}_n) + (\mathbf{I}_n \otimes -\mathbf{P}_{n_s}) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{P}^T_{\varphi(0)} \oplus -\mathbf{P}_0 \\ \mathbf{P}^T_{\varphi(1)} \oplus -\mathbf{P}_1 \\ \vdots \\ \mathbf{P}^T_{\varphi(n_s-1)} \oplus -\mathbf{P}_{n_s-1} \\ \mathbf{P}^T_{\varphi(n_s)} \oplus -\mathbf{P}_{n_s} \end{bmatrix}}_{\mathbf{M}}$$

Define $\Lambda = \{0\} \cup \Omega_{n_s} = \{\lambda_0, \lambda_1, \ldots, \lambda_{n_s}\}$ ordered in the following manner

$$0, \omega^1_{n_s}, \ldots, \omega^{n_s-1}_{n_s}, 1$$

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Inputs: Integers $s \geq 1, k \geq 2$, and $A \in \mathbb{C}^{n\times n}$ for some integer $n \geq 2$.

Outputs: All the $\{k\}$-involutory matrices $\mathbf{R} \in \mathbb{C}^{n\times n}$ such that $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent if any such $\mathbf{R}$ exist.

Step 1 Apply Algorithm 2 to $\mathbf{A}$. If $\mathbf{A}$ is not potentially $\{\mathbf{R}, s+1, k\}$-potent, then no such $\{k\}$-involutory matrix $\mathbf{R}$ exists. Go to End.

Step 2 Compute $\sigma(\mathbf{A})$. Suppose that $\mathbf{A}$ has ℓ distinct eigenvalues. Since $\sigma(\mathbf{A}) \subseteq \Lambda$, there are ℓ indices j_t with $0 \leq j_1 < j_2 < \cdots < j_\ell \leq n_s$ such that $\sigma(\mathbf{A}) = \{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_\ell}\}$.

Step 3 Compute the principal idempotents associated with the eigenvalues of $\mathbf{A}$.

Step 4 Compute $\varphi(j_1), \varphi(j_2), \ldots, \varphi(j_\ell)$.

Step 5 Compute the submatrix $\mathbf{M_A}$ of $\mathbf{M}$ containing only those rows corresponding to eigenvalues of $\mathbf{A}$.

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Step 6 Find the general solution v to $\mathbf{M_A} v = \mathbf{0}$. The $n^2 \times 1$ vector v will depend on $d = \dim(\ker(\mathbf{M_A}))$ arbitrary parameters.

Step 7 If $v = \mathbf{0}$, or equivalently, if $d = 0$, then go to Step 11.

Step 8 Treating v as $v = v(\mathbf{R})$ for an $n \times n$ complex matrix $\mathbf{R}$ containing d parameters, recover $\mathbf{R}$ from v.

Step 9 Determine the allowed values for the d arbitrary parameters so that $\mathbf{R}^k = \mathbf{I}_n$. If there are no allowed parameter values, then go to Step 11.

Step 10 The output is the set of all matrices $\mathbf{R}$ whose parameter values are allowed.

Step 11 "There is no $\{k\}$-involutory matrix $\mathbf{R}$ such that $\mathbf{A}$ is $\{\mathbf{R}, s+1, k\}$-potent."

End

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Example

For $s = 2,\ k = 4$ and

$$\mathbf{A} = \begin{bmatrix} -49i & 40i & -10i \\ 18 - 78i & -15 + 64i & 4 - 16i \\ 72 - 72i & -60 + 60i & 16 - 15i \end{bmatrix},$$

Algorithm 3 provides the solutions

$$\mathbf{R} = \begin{bmatrix} x & y & -\frac{y}{4} \\ t & -\frac{5t}{6} + \frac{5x}{3} + 2y & \frac{2t}{9} - \frac{5y}{6} - \frac{32x}{45} \\ 4t - 6y - \frac{49x}{5} & -\frac{10t}{3} + \frac{32x}{3} + 8y & \frac{8t}{9} - \frac{10y}{3} - \frac{173x}{45} \end{bmatrix}$$

where $x, y, t \in \mathbb{C}$.

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Conclusions

- An algorithm was designed to solve the direct problem related to $\{\mathbf{R}, s+1, k\}$-potent matrices considering $s \geq 1$.
- An algorithm was designed to solve the inverse problem related to $\{\mathbf{R}, s+1, k\}$-potent matrices considering $s \geq 1$.
- The case $s = 0$ can also be treated but, in this case, matrices are not necessarily diagonalizable. So, both diagonalizable and not diagonalizable cases have to be considered separately.

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THANK YOU VERY MUCH !

E-mail: leila.lebtahi@uv.es