On the inverse problem associated to $RA = A^{s+1}R$ when $R^k = I$

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Outline

1. Characterizations of \{R, s + 1, k\}-potent matrices

2. Computing \{R, s + 1, k\}-potent matrices

3. Algorithm for finding matrices \(R\) such that \(A\) is a \(\{R, s + 1, k\}\)-potent

4. Numerical example and conclusions
**Definition of \( \{R, s + 1, k\} \)-potent matrices**

**Definition**

Let \( s \in \{1, 2, 3, \ldots \} \) and \( R \in \mathbb{C}^{n \times n} \) be a \( \{k\} \)-involutory matrix, that is, \( R^k = I \). A matrix \( A \in \mathbb{C}^{n \times n} \) is called \( \{R, s + 1, k\} \)-potent if it satisfies

\[
RA = A^{s+1}R
\]

Extends to:

<table>
<thead>
<tr>
<th>Matrices</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>idempotent: ( A^2 = A )</td>
<td>( R = I, s = 1 )</td>
</tr>
<tr>
<td>( {s + 1} )-potent: ( A^{s+1} = A )</td>
<td>( R = I )</td>
</tr>
<tr>
<td>involutory: ( A^2 = I )</td>
<td>( R = I, s = 2, A ) nonsingular</td>
</tr>
<tr>
<td>centrosymmetric: ( AR = RA )</td>
<td>( R = J ) is an exchange matrix, ( s = 0 )</td>
</tr>
<tr>
<td>mirrorsymmetric: ( AR = RA )</td>
<td>( R = I \oplus J \oplus I ) (on sec. diag.), ( s = 0 )</td>
</tr>
</tbody>
</table>
Two problems related to \{R, s + 1, k\}-potent matrices

Characterizations of \{R, s + 1, k\}-potent matrices:

L-S-T-W, Matrices \( A \) such that \( RA = A^{s+1}R \) when \( R^k = I \),


Construction of a group from \{R, s + 1, k\}-potent matrices:

C-L-S-T, On a matrix group constructed from an \{R, s + 1, k\}-potent matrix,

*Linear Algebra and its Applications* 461 (2014) 200–210

**Direct Problem:** How to construct this class of matrices?

Construction of \{R, s + 1, k\}-potent matrices for a given \( \{k\}\)-involutory matrix \( R \).
Two problems related to \( \{ R, s + 1, k \} \)-potent matrices

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**Direct Problem:** How to construct this class of matrices?

Construction of \( \{ R, s + 1, k \} \)-potent matrices for a given \( \{ k \} \)-involutory matrix \( R \).

**Inverse Problem:** How to compute \( \{ k \} \)-involutory matrices \( R \)?

Design of an algorithm to construct matrices \( R \) such that a given matrix \( A \) is
\( \{ R, s + 1, k \} \)-potent.
Constructing a bijection

Let

- \( S = \{1, 2, \ldots, n_s - 1\} \) where \( n_s = (s + 1)^k - 1 \)
- \( \varphi : S \cup \{0\} \to S \cup \{0\} \)
- \( \varphi(j) := b_j \)
- \( b_j \) is the smallest nonnegative integer such that
  \[ b_j \equiv j(s + 1) \pmod{(n_s)} \]

Then \( \varphi \) is a bijective function.
Characterizations of \( \{R, s + 1, k\} \)-potent matrices ...
Theorem

Let

- \( A \in \mathbb{C}^{n \times n}, \ s \in \{1, 2, 3, \ldots \} \)
- \( R \in \mathbb{C}^{n \times n} \) be a \( \{k\}\)-involutory matrix.

Then the following conditions are equivalent:

(a) \( A \) is \( \{R, s + 1, k\} \)-potent.

(b) \( A \) is diagonalizable, \( \sigma(A) \subseteq \{0\} \cup \Omega_{ns} \), \( RP_0 R^{-1} = P_0 \),

\[
RP_\varphi(j) R^{-1} = P_j \quad \text{where} \ j \in S \quad \text{and} \quad RP_{ns} R^{-1} = P_{ns},
\]

\( P_0, P_1, \ldots, P_{ns} \) are the projectors of the spectral decomposition of \( A \) associated to the eigenvalues \( 0, \omega_{ns}^1, \ldots, \omega_{ns}^{n_s-1}, 1 \), respectively.
Theorem

Let

- \( A \in \mathbb{C}^{n \times n}, \ s \in \{1, 2, 3, \ldots \} \)
- \( R \in \mathbb{C}^{n \times n} \) be a \( \{k\}\)-involutory matrix.

Then the following conditions are equivalent:

(a) \( A \) is \( \{R, s + 1, k\}\)-potent.

(c) \( A^{n_s+1} = A \),

\[
\begin{align*}
RP_0 R^{-1} &= P_0, \\
RP_\varphi(j) R^{-1} &= P_j \\
RP_{n_s} R^{-1} &= P_{n_s}
\end{align*}
\]

where \( j \in S \) and \( P_0, P_1, \ldots, P_{n_s} \) are the projectors of the spectral decomposition of \( A \) associated to the eigenvalues \( 0, \omega_{n_s}^1, \ldots, \omega_{n_s}^{n_s-1}, 1 \), respectively.
in terms of the group inverse of $A$

**Theorem**

Let

- $A \in \mathbb{C}^{n \times n}$, $s \in \{1, 2, 3, \ldots \}$
- $R \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix.

Then the following conditions are equivalent:

(a) $A$ is $\{K, s + 1\}$-potent.

(d) $A^\# = A^{n_s - 1}$,

$$RP_0 R^{-1} = P_0, \quad RP_{\varphi(j)} R^{-1} = P_j$$

where $j \in S$ and

$$RP_{n_s} R^{-1} = P_{n_s},$$

$P_0, P_1, \ldots, P_{n_s}$ are the projectors of the spectral decomposition of $A$ associated to the eigenvalues $0, \omega_1^{n_s}, \ldots, \omega_{n_s}^{n_s - 1}, 1$, respectively.
by using a representation of $A$ of index 1

**Theorem**

Let

- $A \in \mathbb{C}^{n \times n}$, $s \in \{1, 2, 3, \ldots \}$
- $R \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix.

Then the following conditions are equivalent:

(a) $A$ is $\{R, s + 1, k\}$-potent.

(e) there are nonsingular matrices: $P \in \mathbb{C}^{n \times n}$, $C \in \mathbb{C}^{r \times r}$ such that

$$A = P \begin{bmatrix} C & O & O \\ O & O & \end{bmatrix} P^{-1} \quad R = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1}$$

where $r = \text{rank}(A)$, $R_1 \in \mathbb{C}^{r \times r}$ and $R_2 \in \mathbb{C}^{(n-r) \times (n-r)}$ are both $\{k\}$-involutory and $C$ is $\{R_1, s + 1, k\}$-potent.
DIRECT PROBLEM:

Construction of \( \{R, s + 1, k\} \)-potent matrices for a given \( \{k\} \)-involutory matrix \( R \)
Computing $\{R, s + 1, k\}$-potent matrices

### A simplification

The cases $R = I_n$ only provide the well-known results corresponding to $A^{s+1} = A$.

Not interesting!
Computing \( \{ R, s + 1, k \} \)-potent matrices

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The cases \( R = I_n \) only provide the well-known results corresponding to \( A^{s+1} = A \). Not interesting!

Diagonalization of \( R \)

Since \( R \) is \( \{ k \} \)-involutory, there is a nonsingular matrix \( T = \begin{bmatrix} t_1 & \ldots & t_n \end{bmatrix} \) such that

\[
R = T \text{ diag } (\omega_1 I_{r_1}, \ldots, \omega_{\ell-1} I_{r_{\ell-1}}, I_\ell) \ T^{-1}
\]

where \( r_1 + \cdots + r_\ell = n \) and \( \omega_i \in \Omega_k, i = 1, \ldots, \ell. \)
Computing \( \{ R, s + 1, k \} \)-potent matrices

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The cases \( R = I_n \) only provide the well-known results corresponding to \( A^{s+1} = A \). Not interesting!

Diagonalization of \( R \)
Since \( R \) is \( \{ k \} \)-involutory, there is a nonsingular matrix \( T = \begin{bmatrix} t_1 & \ldots & t_n \end{bmatrix} \) such that
\[
R = T \, \text{diag} \left( \omega_1 I_{r_1}, \ldots, \omega_{\ell-1} I_{r_{\ell-1}}, I_\ell \right) \, T^{-1}
\]
where \( r_1 + \cdots + r_\ell = n \) and \( \omega_i \in \Omega_k, i = 1, \ldots, \ell \).

We can assume \( r_1 < r_2 < \cdots < r_\ell \). Otherwise, we compute
- \( \omega_i R \) instead of \( R \), \( i = 1, \ldots, k \) for \( k \) odd
- \( -R \) instead of \( R \), for \( k \) even

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**Diagonalization of \( A \)**

\[
A = S \, \text{diag}(\lambda_1, \ldots, \lambda_n) \, S^{-1}
\]

with

\[\lambda_i \in \{0\} \cup \Omega_{ns}, \; i = 1, \ldots, n, \quad S = \begin{bmatrix} s_1 & \ldots & s_n \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix}\]

**Spectral decomposition of \( A \)**

Denoting \( P_i = s_i y_i^T \) we have

\[
A = \sum_{i=1}^{n} \lambda_i P_i
\]
Computing \( \{ R, s + 1, k \} \)-potent matrices

**Diagonalization of** \( A \)

\[
A = S \, \text{diag}(\lambda_1, \ldots, \lambda_n) \, S^{-1}
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\]

**Spectral decomposition of** \( A \)

Denoting \( P_i = s_i y_i^T \) we have

\[
A = \sum_{i=1}^{n} \lambda_i P_i
\]

**Main idea:**

Construction of the \( s_i \)'s and \( y_i \)'s in terms of the \( t_i \)'s.
Computing $\{R, s + 1, k\}$-potent matrices

Construction of the $s_i$’s and $y_j$’s in terms of the $t_i$’s

Since $RP_{\varphi(j)} = P_j$ must hold, we can choose

$$s_j = Rs_{\varphi(j)} \quad \text{and} \quad R^T y_j = y_{\varphi(j)}$$

for $j \in S$ in order to satisfy the equality $RA = A^{s+1}R$.

Example 1

If $R = TDT^{-1}$ is a $\{4\}$-involutory matrix and $D = \text{diag}(i, -i, 1)$

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<th>Construction of matrix $A$</th>
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<td>$s_1 = s'_{\varphi(j)} = t_1 + t_2 + t_3$</td>
<td>$A = \omega^j P_1 + \omega^\varphi(j) P_2 + P_3$</td>
</tr>
<tr>
<td>$s_2 = s'_j = it_1 - it_2 + t_3$</td>
<td></td>
</tr>
<tr>
<td>$s_3 = t_3$</td>
<td></td>
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where $\omega = \omega(s+1)^4 - 1$ and $j \in S$. 
Computing \( \{R, s + 1, k\}\)-potent matrices

**Example 2**

If \( R = TDT^{-1} \) is a \( \{4\}\)-involutory matrix and \( D = \text{diag}(-1, -1, i, i) \),

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<td>( s_1 = s'_{\varphi(j)} = t_1 + t_3 )</td>
<td>( A = \omega^j P_1 + \omega^{\varphi(j)} P_2 + \omega^a P_3 + \omega^{\varphi(a)} P_4 )</td>
</tr>
<tr>
<td>( s_2 = s'_j = -t_1 + it_3 )</td>
<td></td>
</tr>
<tr>
<td>( s_3 = s'_{\varphi(a)} = t_2 + t_4 )</td>
<td></td>
</tr>
<tr>
<td>( s_4 = s'_a = -t_2 + it_4 )</td>
<td></td>
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where \( \omega = \omega_{(s+1)^4-1}, j \in S \), and moreover \( a \in S - \{j, \varphi(j)\} \) such that \( \varphi(a) \neq a \).
Computing $\{R, s + 1, k\}$-potent matrices

**Example 3**

If $R = TDT^{-1}$ is a $\{4\}$-involutory matrix and $D = \text{diag}(-1, -1, -i, i, 1)$

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<td>$s_1 = s'_{\varphi(j)} = t_1 + t_3 + t_4 + t_5$</td>
<td>$A = \omega^j P_1 + \omega^{\varphi(j)} P_2 + \omega^a P_3 + \omega^{\varphi(a)} P_4 + P_5$</td>
</tr>
<tr>
<td>$s_2 = s'_j = -t_1 - it_3 + it_4 + t_5$</td>
<td></td>
</tr>
<tr>
<td>$s_3 = s'_{\varphi(a)} = t_2$</td>
<td></td>
</tr>
<tr>
<td>$s_4 = s'_a = -t_2$</td>
<td></td>
</tr>
<tr>
<td>$s_5 = s'_b = t_5$</td>
<td></td>
</tr>
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</table>

where $\omega = \omega_{(s+1)^4-1}$, $j \in S$, and moreover $a \in S - \{j, \varphi(j)\}$ such that $\varphi(a) \neq a$ and $b \in S$ such that $\varphi(b) = b$. 
Computing $\{R, s + 1, 2\}$-potent matrices

Algorithm 1

*Inputs:* An involutory matrix $R$.

*Outputs:* A $\{R, s + 1, 2\}$-potent matrix $A \in \mathbb{C}^{n \times n}$ and the projectors $P_i$.

1. **Step 1** Diagonalize $R$ as $R = T \text{ diag } (-I_r, I_{n-r}) T^{-1}$.
2. **Step 2** If $r > n - r$, change $R$ to $-R$ and rearrange as in Step 1.
3. **Step 3** For $i = 1, \ldots, r$, compute $s_{2i-1} = t_i + t_{i+r}$ and $s_{2i} = -t_i + t_{i+r}$.
4. **Step 4** For $i = 2r + 1, \ldots, n$, set $s_i = t_i$.
5. **Step 5** Solve the linear systems $Sy_i = e_i$, where $e_i$ are the canonical basis vectors of $\mathbb{C}^n$ for $i = 1, \ldots, n$.
6. **Step 6** Compute $P_i = s_i y_i^T$ for $i = 1, \ldots, n$.
7. **Step 7** For $i = 1, \ldots, r$, compute $Q_i = \omega P_{2i-1} + \omega^{\varphi(1)} P_{2i}$.
8. **Step 8** Compute $A = \sum_{i=1}^{r} Q_i + \sum_{i=2r+1}^{n} P_i$.

End
Numerical example: an \( \{R, 3, 2\} \)-potent matrix

\[ s = 2, \ k = 2, \ n = 4, \ \mathbf{D} = \text{diag}(-1, -1, 1, 1) \]

\[
\mathbf{R} = \begin{bmatrix}
1.2989 & -1.2069 & 3.5632 & 3.0460 \\
1.3793 & -2.7241 & 4.1379 & 4.8276 \\
-0.2759 & 1.3448 & -3.8276 & -3.9655 \\
0.6437 & -2.1379 & 4.5977 & 5.2529
\end{bmatrix}
\]

\[
\mathbf{A} = \begin{bmatrix}
-0.3820 - 0.0731i & 1.2435 - 0.0853i & -0.9103 + 0.4877i & -0.9834 + 1.1582i \\
0.3657 + 0.5202i & 0.6035 - 0.4145i & -1.0241 - 0.3251i & -1.0180 + 0.4064i \\
-0.2845 - 0.1300i & 0.4145 - 0.7803i & 0.0894 + 0.7884i & -0.6421 + 0.9591i \\
0.3657 + 0.5202i & -0.1036 + 0.2926i & -1.0241 - 0.3251i & -0.3109 - 0.3007i
\end{bmatrix}
\]
INVERSE PROBLEM:

How to compute \( \{k\}\)-involutory matrices \( R \) such that a given matrix \( A \) is \( \{R, s + 1, k\}\)-potent?
Before computing \( \{k\}\)-involutory matrices \( R \)

**Definition**

For a given positive integer \( s \), the square, complex matrix \( A \) is called a *potential \( \{R, s + 1, k\}\)-potent matrix* if \( A^{ns+1} = A \), or equivalently, if \( A \) is diagonalizable and \( \sigma(A) \) is contained in \( \{0\} \cup \Omega_{ns} \).

**Observation**

- \( R \) is completely unspecified here.
- It is generally much easier and faster to test that \( A^{ns+1} = A \) than is to determine the spectrum of \( A \) and to determine that \( A \) is diagonalizable.
Before computing \( \{k\} \)-involutory matrices \( R \)

**Algorithm 2**

*Inputs*: Integers \( s \geq 1, k \geq 2 \), and \( A \in \mathbb{C}^{n \times n} \) for some integer \( n \geq 2 \).

*Output*: A decision on whether \( A \) is potentially \( \{R, s + 1, k\} \)-potent or not.

**Step 1** If either \( A^{ns+1} = A \) or, \( A \) is diagonalizable and \( \sigma(A) \subseteq \{0\} \cup \Omega_{ns} \), then ”\( A \) is potentially \( \{R, s + 1, k\} \)-potent”. Go to End.

**Step 2** ”\( A \) is not potentially \( \{R, s + 1, k\} \)-potent, and there is no \( \{k\} \)-involutory matrix \( R \in \mathbb{C}^{n \times n} \) such that \( A \) is \( \{R, s + 1, k\} \)-potent”.

End
The notations $\otimes$ and $\oplus$ denote the Kronecker product and Kronecker sum of two matrices, respectively.

For any matrix $X = [x_{ij}] \in \mathbb{C}^{n \times n}$, let $v(X) = [v_k] \in \mathbb{C}^{n^2 \times 1}$ be the vector formed by stacking the columns of $X$ into a single column vector. The expression $[v(X)]\{(j-1)n+1,\ldots,(j-1)n+n\}$, for $j = 1, \ldots, n$, denotes the $j^{th}$ column of $X$.

**Property:** If $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ then

$$\text{Ker}(A) \cap \text{Ker}(B) = \text{Ker} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right)$$
We recall that the principal idempotents associated with the eigenvalues $\lambda_1, \ldots, \lambda_\ell$ are given by

$$P_t = \frac{p_t(A)}{p_t(\lambda_t)}$$

where

$$p_t(\eta) = \prod_{i=1, i \neq t}^{\ell} (\eta - \lambda_i)$$

By using the function $\varphi$ and these projectors, it is possible to consider the matrix

$$M = \begin{bmatrix}
P^T_{\varphi(0)} \oplus -P_0 \\
P^T_{\varphi(1)} \oplus -P_1 \\
\vdots \\
P^T_{\varphi(n_s-1)} \oplus -P_{n_s-1} \\
P^T_{\varphi(n_s)} \oplus -P_{n_s}
\end{bmatrix}.$$
Solving the inverse problem: Idea

We focus our attention on solving the matrix equations (in the unknown $R$):

\[ RP_{\varphi(j)} = P_j R \]

that is, to find the common solutions to

\[ RP_{\varphi(j)} = P_j R, \text{ for } j \in S \cup \{0\} \quad \text{and} \quad RP_{n_s} = P_{n_s} R \]

After vectoring, the Kronecker product allows us to write

\[ \nu(RP_{\varphi(j)}) = \nu(P_j R) \iff (P^T_{\varphi(j)} \otimes I_n)\nu(R) = (I_n \otimes P_j)\nu(R), \]

for $j \in S$, and analogously,

\[ \nu(RP_{n_s}) = \nu(P_{n_s} R) \iff (P^T_{n_s} \otimes I_n)\nu(R) = (I_n \otimes P_{n_s})\nu(R). \]
Solving the inverse problem: Idea

By the property about kernels:

we have to find (non trivial) solutions \( v(\mathbf{R}) \) of the null space of

\[
\begin{pmatrix}
(P^T_\varphi(0) \otimes I_n) + (I_n \otimes -P_0) \\
(P^T_\varphi(1) \otimes I_n) + (I_n \otimes -P_1) \\
\vdots \\
(P^T_\varphi(n_s-1) \otimes I_n) + (I_n \otimes -P_{n_s-1}) \\
(P^T_{n_s} \otimes I_n) + (I_n \otimes -P_{n_s})
\end{pmatrix}
\]
Solving the inverse problem: Idea

By the property about kernels:

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\vdots \\
(P^T_{\varphi(n_s-1)} \otimes I_n) + (I_n \otimes -P_{n_s-1}) \\
(P^T_{n_s} \otimes I_n) + (I_n \otimes -P_{n_s})
\end{bmatrix}
= \begin{bmatrix}
P^T_{\varphi(0)} \oplus -P_0 \\
P^T_{\varphi(1)} \oplus -P_1 \\
\vdots \\
P^T_{\varphi(n_s-1)} \oplus -P_{n_s-1} \\
P^T_{\varphi(n_s)} \oplus -P_{n_s}
\end{bmatrix}
\]

\[ M \]
Solving the inverse problem: Idea

By the property about kernels:

we have to find (non trivial) solutions \( v(\mathbf{R}) \) of the null space of

\[
\begin{bmatrix}
(P_{\varphi(0)}^T \otimes I_n) + (I_n \otimes -P_0) \\
(P_{\varphi(1)}^T \otimes I_n) + (I_n \otimes -P_1) \\
\vdots \\
(P_{\varphi(n_s-1)}^T \otimes I_n) + (I_n \otimes -P_{n_s-1}) \\
(P_{n_s}^T \otimes I_n) + (I_n \otimes -P_{n_s})
\end{bmatrix} = \begin{bmatrix}
P_{\varphi(0)}^T \oplus -P_0 \\
P_{\varphi(1)}^T \oplus -P_1 \\
\vdots \\
P_{\varphi(n_s-1)}^T \oplus -P_{n_s-1} \\
P_{n_s}^T \oplus -P_{n_s}
\end{bmatrix} = M
\]

Define \( \Lambda = \{0\} \cup \Omega_{n_s} = \{\lambda_0, \lambda_1, \ldots, \lambda_{n_s}\} \) ordered in the following manner

\[0, \omega_{n_s}^1, \ldots, \omega_{n_s}^{n_s-1}, 1\]
Algorithm 3

**Inputs:** Integers $s \geq 1$, $k \geq 2$, and $A \in \mathbb{C}^{n \times n}$ for some integer $n \geq 2$.

**Outputs:** All the $\{k\}$-involutory matrices $R \in \mathbb{C}^{n \times n}$ such that $A$ is $\{R, s + 1, k\}$-potent if any such $R$ exist.

**Step 1** Apply Algorithm 2 to $A$. If $A$ is not potentially $\{R, s + 1, k\}$-potent, then no such $\{k\}$-involutory matrix $R$ exists. Go to End.

**Step 2** Compute $\sigma(A)$. Suppose that $A$ has $\ell$ distinct eigenvalues. Since $\sigma(A) \subseteq \Lambda$, there are $\ell$ indices $j_t$ with $0 \leq j_1 < j_2 < \cdots < j_\ell \leq n_s$ such that $\sigma(A) = \{\lambda_{j_1}, \lambda_{j_2}, \ldots, \lambda_{j_\ell}\}$.

**Step 3** Compute the principal idempotents associated with the eigenvalues of $A$.

**Step 4** Compute $\varphi(j_1), \varphi(j_2), \ldots, \varphi(j_\ell)$.

**Step 5** Compute the submatrix $M_A$ of $M$ containing only those rows corresponding to eigenvalues of $A$. 
Algorithm 3 (cont.)

Step 6 Find the general solution \( v \) to \( M_A v = 0 \). The \( n^2 \times 1 \) vector \( v \) will depend on \( d = \text{dim}(\ker(M_A)) \) arbitrary parameters.

Step 7 If \( v = 0 \), or equivalently, if \( d = 0 \), then go to Step 11.

Step 8 Treating \( v \) as \( v = v(R) \) for an \( n \times n \) complex matrix \( R \) containing \( d \) parameters, recover \( R \) from \( v \).

Step 9 Determine the allowed values for the \( d \) arbitrary parameters so that \( R^k = I_n \). If there are no allowed parameter values, then go to Step 11.

Step 10 The output is the set of all matrices \( R \) whose parameter values are allowed.

Step 11 "There is no \( \{ k \} \)-involutory matrix \( R \) such that \( A \) is \( \{ R, s + 1, k \} \)-potent."

End
Example

For \( s = 2, \ k = 4 \) and

\[
A = \begin{bmatrix}
-49i & 40i & -10i \\
18 - 78i & -15 + 64i & 4 - 16i \\
72 - 72i & -60 + 60i & 16 - 15i
\end{bmatrix},
\]

Algorithm 3 provides the solutions

\[
R = \begin{bmatrix}
x & y & -\frac{y}{4} \\
t & -\frac{5t}{6} + \frac{5x}{3} + 2y & \frac{2t}{9} - \frac{5y}{6} - \frac{32x}{45} \\
4t - 6y - \frac{49x}{5} & -\frac{10t}{3} + \frac{32x}{3} + 8y & \frac{8t}{9} - \frac{10y}{3} - \frac{173x}{45}
\end{bmatrix}
\]

where \( x, y, t \in \mathbb{C} \).
Conclusions

- An algorithm was designed to solve the direct problem related to \( \{R, s + 1, k\} \)-potent matrices considering \( s \geq 1 \).

- An algorithm was designed to solve the inverse problem related to \( \{R, s + 1, k\} \)-potent matrices considering \( s \geq 1 \).

- The case \( s = 0 \) can also be treated but, in this case, matrices are not necessarily diagonalizable. So, both diagonalizable and not diagonalizable cases have to be considered separately.
THANK YOU VERY MUCH!

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