Numerical methods for large-scale differential matrix equations in control theory

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Outline

- Differential Riccati equations
- The LQR finite-Horizon problem
- Krylov-subspace methods
- Krylov projection + BDF (Backward Differentiation Formulae) for DRE's
- Lyapunov differential matrix equations
- Krylov-BDF and Krylov-exponential approaches
- Some numerical experiments.

Solving large scale differential matrix equations

- Symmetric Differential Riccati equations (continuous case)

$$\dot{X}(t) = A(t)^T X(t) + X(t)A(t) - X(t)B(t)B(t)^T X(t) + C(t)^T C(t).$$

- Differential Lyapunov matrix equations

$$\dot{X}(t) = A(t)X(t) + X(t)A(t)^T + F(t)F(t)^T$$

Application: Control theory, Model reductions, Linear time varying dynamical systems, ...

PART I: Differential Riccati Equations (DREs)

We consider the continuous-time differential Riccati equation (DRE in short) on the time interval $[0, T_f]$ of the form

$$\begin{cases} \dot{X}(t) = A^T X(t) + X(t) A - X(t) B B^T X(t) + C^T C \\ X(0) = X_0, \; t \in [0 \; T_f] \end{cases} \quad (1)$$

where X_0 is some given $n \times n$ matrix, $A \in \mathbb{R}^{n \times n}$ is assumed to be large, sparse and nonsingular, $B \in \mathbb{R}^{n \times s}$ and $C \in \mathbb{R}^{s \times n}$.
The matrices B and C are assumed to have full rank with $s \ll n$.
A, B, and C are time-independent.

The Finite-Horizon LQR problem

The Linear Quadratic Regulator (LQR) problem is a well known design technique in the theory of optimal control. It can be described as follows. For each initial state x_0, find the optimal cost $J(x_0, \hat{u})$ such that:

$$J(x_0, \hat{u}) = \inf_{u} \left\{ \int_0^{T_f} \Big(y(t)^T y(t) + u(t)^T u(t) \Big) \, dt \right\},$$

under the dynamic constrains

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A\,x(t) + B\,u(t), \quad x(0) = x_0. \\ y(t) & = & C\,x(t) \end{array} \right.$$

$x(t)$ is the state vector of dimension n, $u(t) \in \mathbb{R}^p$ the control vector and $y(t)$ the output vector of length s.

Find also the optimal input control $\hat{u}(t)$:

$$J(x_0, \hat{u}) = \int_0^{T_f} \left(\hat{y}(t)^T \, \hat{y}(t) + \hat{u}(t)^T \, \hat{u}(t) \right) \, dt,$$

Assuming that the pair (A, B) is stabilizable (*i.e.* there exists a matrix S such that $A - B\,S$ is stable) and the pair (C, A) is detectable (i.e., (A^T, C^T) stabilizable):

$$\hat{u}(t) = -B^T \, P(t) \, \hat{x}(t),$$

where $P(t) \in \mathbb{R}^{n \times n}$ is the unique solution to the following differential Riccati equation

$$\dot{P} + A^T P + PA - PBB^T P + C^T C = 0; \quad P(T_f) = 0.$$

The optimal cost is given by the following quadratic function of the initial state x_0:

$$J(x_0, \hat{u}) = x_0^T P(0) x_0. \tag{2}$$

We notice that if we set $X(t) = P(T_f - t)$ and $X_0 = 0$, then

$$P(t) = X(T_f - t).$$

Then we recover the solution X to the differential Riccati equation (1) and

$$J(x_0, \hat{u}) = x_0^T X(T_f) x_0.$$

These results are summarized in the following theorem:

Theorem

Assume that the pair (A, B) is stabilizable and the pair (A, C) is detectable. Then, the differential algebraic Riccati equation (1) has a unique positive solution X on $[0,\ T_f]$ and for any initial state x_0, the optimal cost of $J(x_0, u)$ is given by

$$J(x_0, \hat{u}) = x_0^T X(T_f) x_0,$$

where the optimal control is given by

$$\hat{u}(t) = -B^T X(T_f - t)\hat{x}(t),$$

and the optimal trajectory is determined by

$$\dot{\hat{x}}(t) = (A - BB^T X(T_f - t))\hat{x}(t), \ \text{with } \hat{x}(0) = x_0.$$

Low rank solutions for matrix Riccati equations!!

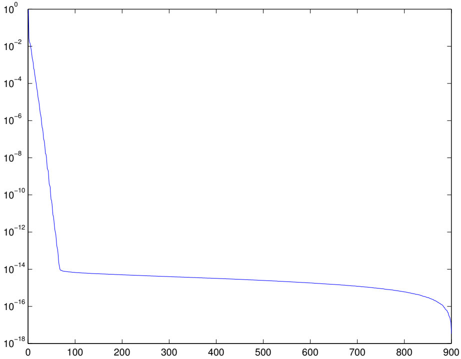

FIGURE – A=fdm900. Singular values of the exact stabilizing solution

The singular values **decay rapidaly to zero** and this suggests to seek for methods producing approximate solutions having a **low rank**.

Existing methods for large algebraic and differential cases

- Algebraic Riccati equations
  - Block Arnoldi : [Jaimoukha(98); J. (03); Simoncini & Szyld (12,16)]
  - Newton-LR-ADI:[Benner & al. 98, 05, 14]
  - Global Arnoldi : [J. (05)]
  - Extended Block Arnoldi: [Heyouni & J. (08)]
  - Direct methods: [D. Bini et al. (12)]
  - EBA for Nonsymmetric CAREs : [Bentbib, J. & Sadek (15)]
  - Extrapolation methods :[El Moallem & Sadok(14)]
- **Differential Riccati equations**
  - LR-ADI-Based methods: [Benner Mena & Stillfjord (15-17)]
  - EBA for Diff. Riccati equations: [Hached & J. (16)].
  - Expo-Krylov for Diff. Lyapunov equations: [Hached & J. (17)]
  - The expon-Hamiltonian approach: [J. (18)].

The extended block Arnoldi algorithm

The extended block-Krylov subspace $\mathcal{K}_m(A, V)$ of $\mathbb{R}^n$ which is considered here is:

$$\mathcal{K}_m(A, V) = Range([V, A^{-1} V, A V, A^{-2} V, A^2 V, \ldots, A^{-m} V, A^{m-1} V]).$$

Notice that the subspace $\mathcal{K}_m(A, V)$ is a **sum of two block Krylov subspaces**

$$\mathcal{K}_m(A, V) = \mathbb{K}_m(A, V) + \mathbb{K}_m(A^{-1}, A^{-1} V)$$

where $\mathbb{K}_m(A, V) = Range([V, A V, \ldots, A^{m-1} V])$ and $V \in \mathbb{R}^{n \times s}$.

The Extended Block Arnoldi (EBA) builds an orthonormal basis $\mathcal{V}_m$ of the Krylov subspace $\mathrm{Range}(V, A\,V, \ldots, A^{m-1}\,V, A^{-1}\,V, \ldots, (A^{-1})^m\,V)$.

Let $\mathcal{T}_m = \mathcal{V}_m^T\,A\,\mathcal{V}_m$ and set $\bar{\mathcal{T}}_m = \mathcal{V}_{m+1}^T\,F\,\mathcal{V}_m$, then we have

$$\begin{aligned} A\,\mathcal{V}_m &= \mathcal{V}_{m+1}\,\bar{\mathcal{T}}_m, \\ &= \mathcal{V}_m\,\mathcal{T}_m + V_{m+1}\,T_{m+1,m}\,E_m^T. \end{aligned}$$

$E_m = [O_{2s\times 2(m-1)s}, I_{2s}]^T$ is the matrix of the last $2s$ columns of the $2ms \times 2ms$ identity matrix I_{2ms}.

The BDF method for solving DREs

The BDF approximation X_{k+1} of $X(t_{k+1})$ is given by the implicit relation

$$X_{k+1} = \sum_{i=0}^{p-1} \alpha_i X_{k-i} + h\beta \mathcal{F}(X_{k+1}),$$

where $h = t_{k+1} - t_k$ is the step size, α_i and β_i are the coefficients of the BDF method as listed in Table 2 and $\mathcal{F}(X)$ is given by

$$\mathcal{F}(X) = A^T X + X A - X B B^T X + C^T C.$$

p	β	α_0	α_1	α_2
1	1	1		
2	2/3	4/3	-1/3	
3	6/11	18/11	-9/11	2/11

TABLE – Coefficients of the p-step BDF method with $p \leq 3$.

Then, X_{k+1} solves the following matrix equation

$$-X_{k+1} + h\beta(C^T C + A^T X_{k+1} + X_{k+1}A) - X_{k+1}BB^T X_{k+1} + \sum_{i=0}^{p-1} \alpha_i X_{k-i} = 0,$$

which can be written as the following continuous-time algebraic Riccati equation

$$\mathcal{A}^T X_{k+1} + \mathcal{A} X_{k+1} - X_{k+1} \mathcal{B} \mathcal{B}^T X_{k+1} + \mathcal{C}_{k+1}^T \mathcal{C}_{k+1} = 0,$$

assuming that at each timestep, X_k can be factorized as a low rank product $X_k \approx Z_k Z_k^T$, $Z_k \in \mathbb{R}^{n \times m_k}$, with $m_k \ll n$, the coefficients matrices are given by

$$\mathcal{A} = h\beta A - \frac{1}{2} I, \;\; \mathcal{B} = \sqrt{h\beta} B; \mathcal{C}_{k+1} = [\sqrt{h\beta} C, \sqrt{\alpha_0} Z_k^T, \ldots, \sqrt{\alpha_{p-1}} Z_{k+1-p}^T]^T.$$

The First approach: BDF+Newton or EBA method

After using the PDF integration formulae, we solve the obtained Algebraic Riccati equation and we can use:

- Direct solvers: `care` with matlab for moderate sizes ($n < 1000$)
- For large problems: Inexact Newton-Kleinman's method combined with an iterative method (EBA or LR-ADI) for the numerical resolution of large-scale Lyapunov equations.
- Using EBA for large-scale algebraic Riccati equations [Heyouni & J. 09]

When using the Kleinman-Newton method, we obtain the following algorithm

We define a sequence of approximates to X_{k+1} as follows:

- Set $X_{k+1}^0 = X_k$
- Build the sequence $\left(X_{k+1}^p\right)_{p\in\mathbb{N}}$ defined by

$$X_{k+1}^{p+1} = X_{k+1}^p - \mathcal{F}_{X_{k+1}^p}(F(X_{k+1}^p)$$

where the Fréchet derivative $\mathcal{F}$ of F at X_{k+1}^p is given by

$$\mathcal{F}_{X_{k+1}^p}(H) = (\mathcal{A} - \mathcal{B}\,\mathcal{B}^T\,X_{k+1}^p)^T\,H \,+\, H\,(\mathcal{A} - \mathcal{B}\,\mathcal{B}^T\,X_{k+1}^p)$$

X_{k+1}^{p+1} is the solution to the Lyapunov equation

$$(\mathcal{A}-\mathcal{B}\,\mathcal{B}^T\,X_{k+1}^p)^T\,X + X\,(\mathcal{A}-\mathcal{B}\,\mathcal{B}^T\,X_{k+1}^p) + X_{k+1}^p\,\mathcal{B}\mathcal{B}^T\,X_{k+1}^p + \mathcal{C}_{k+1}^T\mathcal{C}_{k+1} \;= 0.$$

and we apply Extended-Block-Arnoldi (EBA) to the last Lyapunov matrix equation.

The second approach: Projecting and solving the low dimensional DRE

Apply the Extended Block Arnoldi (EBA) algorithm to the pair (A^T, C^T) and get the orthonormal matrix $\mathcal{V}_m$ and the block-Hessenberg matrix $\mathcal{T}_m = \mathcal{V}_m^T A^T \mathcal{V}_m$.
Let $X_m(t)$ be the desired low-rank approximate solution to (1) given as

$$X_m(t) = \mathcal{V}_m Y_m(t) \mathcal{V}_m^T,$$

and satisfying the Petrov-Galerkin orthogonality condition

$$\mathcal{V}_m^T R_m(t) \mathcal{V}_m = 0,$$

where $R_m(t)$ is the residual

$$R_m(t) = \dot{X}_m(t) - A^T X_m(t) - X_m(t) A + X_m(t) B B^T X_m(t) - C^T C$$

Then, from (17) and (17), we obtain the low dimensional differential Riccati equation

$$\dot{Y}_m - \mathcal{T}_m\, Y_m - Y_m\, \mathcal{T}_m^T + Y_m\, B_m\, B_m^T\, Y_m \; - C_m^T\, C_m = 0.$$

- We use the BDF method for this **low-dimensional differential Riccati equation**.
- At each time-step of the BDF process, we get a low-order algebraic Riccati equation that is solved by a direct method (`care`).
- To stop the iterations without computing the intermediade approximations, we can use the result of the following theorem

Theorem

Let $X_m(t) = \mathcal{V}_m Y_m(t) \mathcal{V}_m^T$ be the approximation obtained at step m by the Extended Block Arnoldi-BDF(p) method and Y_m solves the low-dimensional differential Riccati equation (18), Then the residual R_m satisfies

$$\| R_m(t) \| = \| T_{m+1,m} \hat{Y}_m(t) \|,$$

where $\hat{Y}_m$ is the $2s \times 2ms$ matrix corresponding to the last $2s$ rows of Y_m.

- The approximate solution is not required at each iteration
- We compute it only when convergence is achieved and in a **factored form** (save storage and useful for applications) as follows:

Consider the singular value decomposition of the matrix

$$Y_m(t) = U \Sigma U^T$$

where Σ is the diagonal matrix of the singular values of Y_m sorted in decreasing order.

Let U_l be the $2m \times l$ matrix of the first l columns of U corresponding to the l singular values of magnitude greater than some tolerance $dtol$. We obtain the truncated singular value decomposition

$$Y_m(t) \approx U_l \Sigma_l U_l^T$$

where $\Sigma_l = \mathrm{diag}[\sigma_1, \ldots, \sigma_l]$. Setting $Z_m(t) = \mathcal{V}_k U_l \Sigma_l^{1/2}$, it follows that

$$X_m(t) \approx Z_m(t) Z_m(t)^T.$$

The following result shows that the approximation X_m is an exact solution of a perturbed differential Riccati equation.

Theorem

Let X_m be the approximate solution given by (17). Then we have

$$\dot{X}_m(t) = (A - F_m)^T X_m + X_m (A - F_m) - X_m B B^T X_m + C^T C.$$

where $F_m = V_m T_{m+1,m}^T V_{m+1}^T$.

For the error, we have

Theorem

Let X be a solution of (1) and let X_m be the approximate solution obtained at step m. The error $E_m = X - X_m$ satisfies the following differential Riccati equation

$$\dot{E}_m(t) = (A^T - XBB^T)E_m + E_m(A - BB^T X) + E_m BB^T E_m + R_m,$$

The projected Finite-horizon LQR problem

Consider the LTI system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A\,x(t) + B\,u(t), \quad x(0) = x_0. \\ y(t) & = & C\,x(t) \end{array} \right.$$

and the projected low order dynamical system (onto the extended block Krylov subspace $\mathcal{K}_m(A^T, C^T)$):

$$\left\{ \begin{array}{rcl} \dot{\tilde{x}}_m(t) & = & \mathcal{T}_m\,\tilde{x}_m(t) + B_m\,\tilde{u}_m(t), \quad \tilde{x}_m(0) = x_{m,0}. \\ \tilde{y}_m(t) & = & C_m\,\tilde{x}_m(t) \end{array} \right. \quad (3)$$

where $B_m = \mathcal{V}_m^T B$, $C_m^T = \mathcal{V}_m^T C^T = \mathcal{E}_1\,\Lambda_{1,1}$ and $x_{m,0} = \mathcal{V}_m^T x_0$.
For small values of the iteration number m,
$x_m(t) = \mathcal{V}_m \tilde{x}_m(t)$ is an approximation of the original large state $x(t)$.

Theorem

Assume at step m that $(\mathcal{T}_m, B_m)$ *is stabilizable and that* $(C_m, \mathcal{T}_m)$ *is detectable and consider the low dimension LQR problem with finite time-horizon:*
Minimize

$$J_m(x_{m,0}, \tilde{u}_m) = \int_0^{T_f} \left(\tilde{y}_m(t)^T\, \tilde{y}_m(t) + \tilde{u}_m(t)^T\, \tilde{u}_m(t)\right)\, dt, \qquad (4)$$

under the dynamic constrains (3). *Then, the unique optimal feedback* $\tilde{u}_{*,m}$ *minimizing the cost function* (4) *is given by*

$$\tilde{u}_{*,m}(t) = -B_m \widetilde{Y}_m(t) \hat{x}_m(t),$$

where $\widetilde{Y}_m(t) = Y_m(T_f - t)$ *and* Y_m *is the unique stabilizing solution of the corresponding low-dimensional differential Riccati equation..*

The optimal projected state satisfies $\dot{\tilde{x}}_m(t) = (A - BB^T \widetilde{Y}_m(t))\tilde{x}_m(t)$ and the optimal cost is given by the following quadratic function of the initial state x_0

$$\begin{aligned} J_m(x_{m,0}, \hat{u}_m) &= x_{m,0}^T \widetilde{Y}_m(0) x_{m,0} = x_0 \mathcal{V}_m \widetilde{Y}_m(0) \mathcal{V}_m^T x_0 && (5) \\ &= x_0 P_m(0) x_0 = x_0^T X_m(T_f) x_0, && (6) \end{aligned}$$

This shows clearly that the reduced optimal cost is an approximation of the initial optimal cost.

Part II: Large-scale differential Lyapunov matrix equations

We consider the Lyapunov differential matrix equation (DRE in short) of the form

$$\dot{X}(t) = A(t)\,X(t) + X(t)\,A^T(t) + B(t)B(t)^T$$

and $X(t_0) = X_0, \quad t \in [t_0,\ T_f]$,
This problem is equivalent to the following

$$\dot{x}(t) = \mathcal{A}(t)x(t) + b(t), \tag{7}$$

where $\mathcal{A} = I \otimes A(t) + A(t) \otimes I$, $x(t) = vec(X(t))$.
Then one can use an integration method to solve (7) . However, solving numerically the problem (7) is not recommended for large problems.

Theorem

The unique solution of the general Lyapunov differencial equation

$$\dot{X}(t) = A(t)\,X + X\,A(t)^T + M(t);\;\; X(0) = X_0$$

is defined by

$$X(t) = \Phi_A(t,t_0)X_0\Phi_A^T(t,t_0) + \int_{t_0}^{t} \Phi_A(t,\tau)M(\tau)\Phi_A^T(t,\tau)d\tau.$$

where the transition matrix $\Phi_A(t,t_0)$ *is the unique solution of the problem*

$$\dot{\Phi}_A(t,t_0) = A(t)\Phi_A(t,t_0),\;\; \Phi_A(t_0,t_0) = I.$$

Futhermore, if A *is constant, then*

$$X(t) = e^{(t-t_0)A}X_0e^{(t-t_0)A^T} + \int_{t_0}^{t} e^{(t-\tau)A}M(\tau)e^{(t-\tau)A^T}d\tau,\;\; t\in\mathbb{R}.$$

Projecting onto a Krylov subspace

We first apply the extended block Arnoldi algorithm to the pair (A, B) to get the matrices $\mathcal{V}_m$ and $\bar{\mathcal{T}}_m$; A is assumed to be time-invariant!!
Let $X_m(t)$ be the desired approximate solution given as

$$X_m(t) = \mathcal{V}_m Y_m(t) \mathcal{V}_m^T,$$

satisfying the Petrov-Galerkin orthogonality condition

$$\mathcal{V}_m^T R_m(t) \mathcal{V}_m = 0,$$

We obtain the low dimensional differential Lyapunov equation

$$\dot{Y}_m(t) - \mathcal{T}_m Y_m - Y_m \mathcal{T}_m^T - B_m B_m^T = 0,$$

solved by the BDF integration method.

Theorem

The error $E_m = X - X_m$ *satisfies the following equation*

$$\dot{E}_m(t) = AE_m(t) + E_m(t)A^T + R_m(t), \tag{8}$$

where $R_m(t)$ *is the residual given by*
$R_m(t) = \dot{X}_m(t) - A\,X_m(t) - X_m(t)\,A^T - BB^T.$

Theorem

Assume that A *is stable matrix and* $X(0) = X_m(0)$. *Then we have the following upper bound*

$$\| \, E_m(t) \, \| \leq (t - t_0) \; \| \, R_m \, \|_\infty \; e^{2\mu_2(A)t},$$

where $\mu_2(A) = \dfrac{1}{2}\lambda_{max}(A + A^T)$

A second approach: using an approximation of the matrix exponential

We give a second approach for computing approximate solutions to large-scale Lyapunov differential equations. Coming to the expression of the exact solution as

$$X(t) = e^{(t-t_0)A} X_0 e^{(t-t_0)A^T} + \int_{t_0}^{t} e^{(t-\tau)A} BB^T e^{(t-\tau)A^T} \, d\tau, \; t \in \mathbb{R}. \quad (9)$$

We can see that one possibility for approximating $X(t)$, is to approximate the term $e^{(t-\tau)A}B$ and then use a quadrature method to compute the desired approximate solution.

Let $\mathcal{V}_m = [V_1, \ldots, V_m]$ be the matrix whose block columns are obtained by the extended block Arnoldi algorithm. Then, an approximation to $Z = e^{(t-\tau)A} B$ can be obtained as

$$Z_m(t) = \mathcal{V}_m e^{(t-\tau)\mathcal{T}_m} \mathcal{V}_m^T B$$

where $\mathcal{T}_m = \mathcal{V}_m A \mathcal{V}_m^T$ and $\mathcal{V}_m^T B = \mathcal{E}_1 \Lambda_{1,1}$(QR-decomposition). Therefore, the term appearing in the integral expression can be approximated as

$$e^{(t-\tau)A} B B^T e^{(t-\tau)A^T} \approx Z_m(t) Z_m(t)^T.$$

If for simplicity we set $X_0 = 0$, an approximation to the solution of the differential Lyapunov equation can be expressed as

$$X_m(t) = \mathcal{V}_m G_m(t)(\mathcal{V}_m)^T,$$

where

$$G_m(t) = \int_{t_0}^{t} \widetilde{G}_m(\tau)\widetilde{G}_m^T(\tau)d\tau, \tag{10}$$

and $\widetilde{G}_m(\tau) = e^{(t-\tau)\mathcal{T}_m}\mathcal{E}_1 \Lambda_{1,1}$.

- Notice that we have

$$\mathcal{V}_m^T R_m(t)(\mathcal{V}_m) = 0$$

- For the practical computation of $e^{(t-\tau)\mathcal{T}_m}$ we can use the 'scaling and squaring method', by Higham `expm`.
- $G_m(t)$ can be computed using a quafrature formulae.

We can also state the following result.

Theorem

Let $G_m(t)$ as defined by (10), then it satisfies the following low-order differential Lyapunov matrix equation

$$\dot{G}_m(t) = \mathcal{T}_m G_m(t) + G_m(t)\mathcal{T}_m + \widetilde{C}_m \widetilde{C}_m^T$$

- $G_m(t)$ could be computed as a solution of a differential Lyapunov matrix equation (which is also the first approach)
- OR by a matrix exponential approximation + a quadrature formulation.
- The two approches are theoreticaly equivalent.

Balanced truncation for linear time-varying dynamical systems

Consider the linear-time varying (LTV) dynamical systems

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t), \; x(0) = 0, \\ y(t) & = & C(t)x(t), \end{array} \right. \tag{11}$$

The LTV dynamical system (11) can also be denoted as

$$\Sigma(t) \equiv \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & 0 \end{array} \right].$$

The reduced order LTV dynamical system can be stated as follows

$$\Sigma_m \left\{ \begin{array}{l} \dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)u(t) \\ \\ y_m(t) = C_m(t)x_m(t) \end{array} \right.$$

where $x_m \in \mathbb{R}^m$, $y_m \in \mathbb{R}^s$, $A_m \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times p}$ and $C_m \in \mathbb{R}^{s\times m}$ with $m \ll n$.

Problem: : How to choose the reduced order model?; A_m ...

The reduced order dynamical system should be constructed such that

- the norm $\|y - y_m\|$ should be computable and close to zero.
- Some properties of the original system such as stability should be preserved.
- The method should be efficient for large problems.

Methods: two classes

- Balanced truncation (solve large diff Lyapunov or diff Riccati equations)
- Moment matching (use Krylov based methods: Arnoldi-type, Lanczos-type,...)

One of the well known methods for contructing such reduced-order dynamical systems is the balanced truncation method for LTV systems.

This method requires the LTV controllability and observability Gramians $P(t)$ and $Q(t)$ defined as the solutions of the differential Lyapunov matrix equations

$$\dot{P}(t) = A(t)P(t) + P(t)A(t)^T + B(t)B(t)^T, \; P(0) = 0,$$

and

$$\dot{Q}(t) = A^T(t)P(t) + P(t)A(t) + C(t)^T C(t), \; Q(T_f) = 0.$$

Let us see now how to construct the low order model. Consider the Cholesky decompositions of the Gramians $P(t)$ and $Q(t)$:

$$P(t) = L_c L_c^T, \;\; Q(t) = L_o(t) L_o(t)^T,$$

and the singular value decomposition of $L_c(t)^T L_o(t)$ as

$$L_c(t)^T L_o(t) = \mathcal{Z}(t)\Sigma(t)\mathcal{Y}(t)^T.$$

We set

$$V_m(t) = L_o(t)\mathcal{Y}_m(t)\Sigma_m(t)^{-1/2} \text{ and } W_m(t) = L_c(t)\mathcal{Z}_m(t)\Sigma_m(t)^{-1/2},$$

where $\Sigma_m(t) = diag(\sigma_1(t), \ldots, \sigma_m(t))$; $\mathcal{Z}_m(t)$ and $\mathcal{Y}_m(t)$ correspond to the leading m columns of the matrices $\mathcal{Z}(t)$ and $\mathcal{Y}(t)$.

The matrices A_m, B_m and C_m of the reduced LTV system are such that

$$W_m(t)^T V_m(t) A_m(t) = V_m(t)^T A(t) W_m(t) - V_m(t)^T \dot{W}_m(t),$$

and

$$B_m(t) = V_m(t)^T B(t),\ C_m(t) = C(t) W_m(t).$$

The state $x(t)$ is approximated by $\tilde{x}(t) = V_m(t) x_m(t)$.

For large problems, instead of using the Cholesky factors, it is more convenient to use the low-rank decompositions

$$P(t) \approx Z_{m,p}(t)Z_{m,p}^T(t), \quad Q(t) \approx Z_{m,q}(t)Z_{m,q}^T(t).$$

- Another possibility is to use directly the matrix-basis of the extended block Arnoldi to construct the reduced order model (without solving differential Lyapunov equations)
- We have no formulation for the upper bounds of the errors!!!
- Using trnasfer functions!!

Numerical Examples

We reported the results given by the following approaches :

- EBA-PDF: EBA+ BDF to the projected problem.
- EBA-EXP: EBA-EXP + quadrature formulae, for diff. Lyapunov Eqs.

Example 1. The matrix A was obtained from the 5-point discretization of the operators

$$L_A = \Delta u - f_1(x,y)\frac{\partial u}{\partial x} + f_2(x,y)\frac{\partial u}{\partial y} + g_1(x,y),$$

on the unit square $[0,1] \times [0,1]$ with homogeneous Dirichlet boundary conditions. The number of inner grid points in each direction is n_0 and the dimension of the matrix A was $n = n_0^2$.

We set $f_1(x,y) = 10xy$, $f_2(x,y) = e^{x^2 y}$, $f_3(x,y) = 100y$, $f_4(x,y) = x^2 y$, $g_1(x,y) = 20xy$ and $g_2(x,y) = x\,y$.

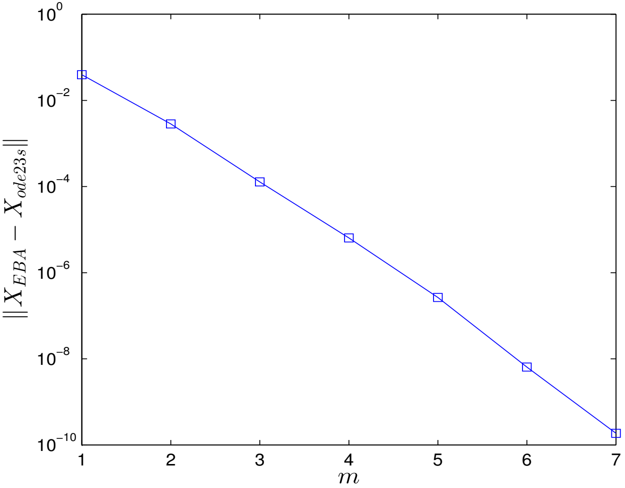

FIGURE – Norms of the errors $\|X_{EBA}(T_f) - X_{ode23s}(T_f)\|_F$.

size(A)	ode23s	EBA+BDF(2)	BDF(2)+Newton-EBA
49×49	30.2	6.5	37.1
100×100	929.3	12.3	41.6
900×900	$--$	45.2	926.1

TABLE – runtimes for ode23s, EBA+BDF(2) and BDF(2)+Newton

size(A)	EBA-BDF(2)	Relative residual norms
22500×22500	77.9 s	$\mathcal{O}(10^{-9})$ ($m = 24$)
160000×160000	283.4 s	$\mathcal{O}(10^{-8})$ ($m = 33$)

TABLE – runtimes and relative residual norms for EBA+BDF(2)

The next figure shows the norm of the residual R_m versus the number m of Extended Block Arnoldi iterations for the $n = 6400$.

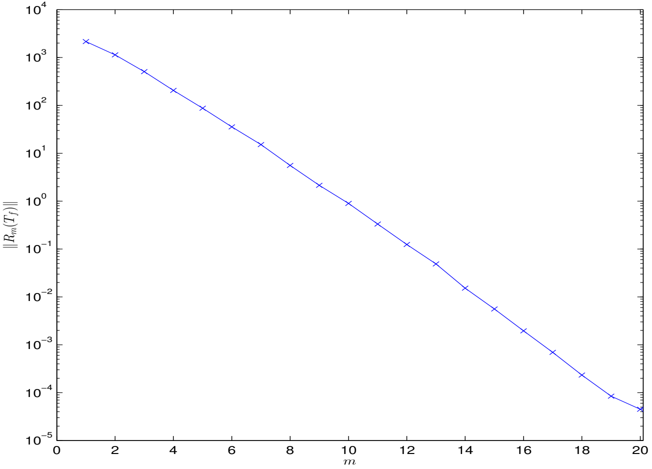

FIGURE – Residual norms $\|R_m(T_f)\|$ versus number of Extended-Arnoldi

Example 2 This example comes from the autonomous linear-quadratic optimal control problem of one dimensional heat flow

$$\begin{aligned} \frac{\partial}{\partial t} x(t,\eta) &= \frac{\partial^2}{\partial \eta^2} x(t,\eta) + b(\eta)u(t) \\ x(t,0) &= x(t,1) = 0, t > 0 \\ x(0,\eta) &= x_0(\eta), \eta \in [0,1] \\ y(x) &= \int_0^1 c(\eta)x(t,\eta)d\eta, x > 0. \end{aligned}$$

Using a standard finite element approach based on the first order B-splines, we obtain the following ordinary differential equation

$$\left\{ \begin{array}{lcl} M\dot{\mathrm{x}}(t) & = & K\mathrm{x}(t) + Fu(t) \\ y(t) & = & C\mathrm{x}(t), \end{array} \right. \tag{12}$$

where the matrices M and K are given by:

$$M = \frac{1}{6n} \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}, K = -\alpha\, n \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

Using the semi-implicit Euler method, we get the following discrete dynamical system

$$(M - K)\,\dot{x}(t) = M\,x(t) + \,Fu_k.$$

We set $A = -(M - \Delta tK)^{-1}\,M$ and $B = \Delta t\,(M - \Delta tK)^{-1}\,F$.

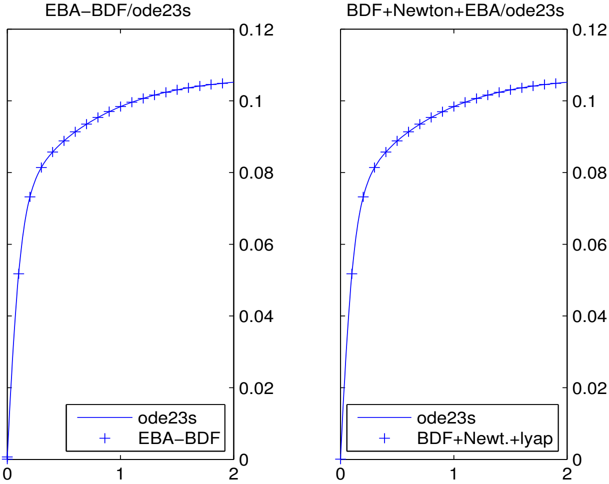

FIGURE – Computed values of $X_{11}(t)$ versus the time $t \in [0, 2]$ for ode23s and EBA-BDF (left) and for ode23s with BDF(2)+Newton+EBA (right)

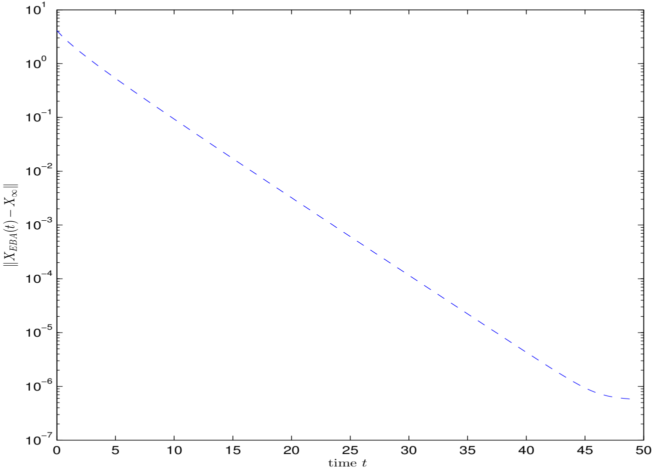

FIGURE – Error norms $\| X_{EBA-BDF}(t) - X_\infty \|$ for $t \in [0, 50]$.

size(A)	Times (s)	Residual norms	Iterations (m)
4900	$64s$	7.62×10^{-10}	14
6400	$163s$	1.43×10^{-10}	19
10000	$457s$	2.35×10^{-10}	21

TABLE – EBA-BDF(2) method: times (s), residual norms, number of EBA iterations (m).

A test for Diff. Lyapunov equations

Example 3

The matrices were extracted from the IMTEK collection [1]. We compared the EBA-BDF(2) method to the EBA-exp method for all the available problem sizes $n = 20209$ and $n = 79841$, on the time interval $[0\,,1000]$.

size(A)	EBA-exp	EBA-BDF(2)	Relative residual norms
20209×20209	57.8 s	231 s	$\mathcal{O}(10^{-7})$
79841×79841	448 s	722 s	$\mathcal{O}(10^{-7})$

TABLE – Optimal Cooling of Steel Profiles: runtimes and residual norms for EBA-exp and EBA-BDF(2)

1. https://portal.uni-freiburg.de/imteksimulation/downloads/benchmark

THANK YOU