Sharp bounds for eigenvalues of the generalized k, m-step Fibonacci matrices

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Outline

1. Perron-Frobenius Theory Preliminaries for nonnegative matrices
2. Sharp bounds for the Perron root of a nonnegative matrix
3. Application to Generalized Fibonacci matrices

Special classes of nonnegative matrices

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a $n \times n$ real matrix.

- We refer to A as *nonnegative* or *positive* if it is entrywise nonnegative or positive, denoted by writing $A \geq 0$ or $A > 0$.
- We call $A \geq 0$ *irreducible* if and only if $(I + A)^{n-1} > 0$.
- $A \geq 0$ is called *primitive* if and only if $A^m > 0$ for some integer m.
- Equivalently, A is primitive if and only if $A^{(n-1)^2+1} > 0$.
- We define the *spectral radius* of A by

$$\rho(A) = \max \{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A, that is, the set of eigenvalues of A.

Perron-Frobenius Theorem for nonnegative matrices

The main clauses of Perron-Frobenius Theorem for nonnegative matrices [Oskar Perron (1907) and Georg Frobenius (1912)]:

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$ *and* $A \geq 0$. *Then*

- $\rho(A) \in \sigma(A)$ *is a nonnegative eigenvalue of A, known as the Perron root.*
- *The Perron root* $\rho(A)$ *satisfies the inequalities:*

$$\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij}.$$

- *If A is irreducible, then* $\rho(A)$ *is a positive and simple eigenvalue, namely, it has algebraic and geometric multiplicity 1.*
- *If A is primitive, then all other eigenvalues* $\lambda \in \sigma(A)$ *satisfy* $0 < |\lambda| < \rho(A)$.

Sharp bounds for $\rho(A)$ part I

Let $A = [a_{ij}] \in \mathcal{M}_n(\mathbb{R})$, $A \geq 0$ and $r_i = \sum_{j=1}^{n} a_{ij}$ be the i-th row sum, $R_i = r_i - a_{ii}$ be the i-th deleted row sum.

1. Due to Frobenius (1912),

$$\min_{1\leq i\leq n} r_i \leq \rho(A) \leq \max_{1\leq i\leq n} r_i. \tag{1}$$

2. Due to A. Brauer, I.C. Gentry (1974),

$$\frac{1}{2} \min_{\substack{1\leq i,j\leq n \\ i\neq j}} \sigma_{ij} \leq \rho(A) \leq \frac{1}{2} \max_{\substack{1\leq i,j\leq n \\ i\neq j}} \sigma_{ij}, \tag{2}$$

where $\sigma_{ij} = a_{ii} + a_{jj} + \sqrt{(a_{ii} - a_{jj})^2 + 4R_iR_j}$.

3. Due to A. Melman (2013),

$$\frac{1}{2} \min_{1\leq i\leq n} \max_{j\neq i} \tau_{ij} \leq \rho(A) \leq \frac{1}{2} \max_{1\leq i\leq n} \min_{j\neq i} \tau_{ij}, \tag{3}$$

where $\tau_{ij} = a_{ii} + a_{jj} - a_{ij} + R_i + \sqrt{(a_{ii} - a_{jj} - a_{ij} + R_i)^2 + 4a_{ij}R_j}$.

Sharp bounds for $\rho(A)$ part II

Let $A \geq 0$ and $r_1 \geq \cdots \geq r_n$, where r_i is the i-th row sum for $i = 1, \ldots, n$.

4. Due to X. Duan, B. Zhou (2013),

$$\phi_n \leq \rho(A) \leq \min\{\Phi_l : 1 \leq l \leq n\}, \tag{4}$$

where for $S = \min_i a_{ii}$, $T = \min_{i \neq j} a_{ij}$, $M = \max_i a_{ii}$, $N = \max_{i \neq j} a_{ij} > 0$

$$\phi_n = \frac{r_n + S - T + \sqrt{(r_n - S + T)^2 + 4T \sum_{i=1}^{n-1}(r_i - r_n)}}{2},$$

$$\Phi_l = \frac{r_l + M - N + \sqrt{(r_l - M + N)^2 + 4N \sum_{i=1}^{l-1}(r_i - r_l)}}{2}.$$

- If $l = 1$, $\Phi_l = r_1$ and if $T = 0$, $\phi_n = r_n$, hence (4) yields (1), namely,

$$r_n \leq \rho(A) \leq r_1.$$

Sharp bounds for $\rho(A)$ part III

Let $A \geq 0$ with $r_i > 0$, $1 \leq i \leq n$ and let $m_i = \sum_{j=1}^{n} a_{ij} \frac{r_j}{r_i}$ be the i-th average 2-row sum of A with $m_1 \geq \cdots \geq m_n$.

5. Due to R. Xing, B. Zhou (2014),

$$\chi_n \leq \rho(A) \leq \min \{X_l : l = 1, \ldots, n\}, \tag{5}$$

where for the previously defined quantities $S, T, M, N > 0$, and $c = \min_{i,j} \frac{r_j}{r_i}$, $b = \max_{i,j} \frac{r_j}{r_i}$,

$$\chi_n = \frac{m_n + S - cT + \sqrt{(m_n - S + cT)^2 + 4cT \sum_{i=1}^{n-1}(m_i - m_n)}}{2},$$

$$X_l = \frac{m_l + M - bN + \sqrt{(m_l - M + bN)^2 + 4bN \sum_{i=1}^{l-1}(m_i - m_l)}}{2}.$$

- If $l = 1$, $X_l = m_1$ and if $T = 0$, $\chi_n = m_n$, hence (5) yields

$$m_n \leq \rho(A) \leq m_1, \qquad \text{[H. Minc(1988)].}$$

Sharp bounds for $\rho(A)$ part IV

Let $A \geq 0$ with $r_i > 0$, $1 \leq i \leq n$ and let $s_i = \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{n} a_{jk} \frac{r_k}{r_i}$ be the i-th average 3-row sum of A with $s_1 \geq \cdots \geq s_n$.

6 Due to H. Lin, B. Zhou (2017),

$$\sqrt{\psi_n} \leq \rho(A) \leq \min\left\{\sqrt{\Psi_l} : l = 1, \ldots, n\right\}, \tag{6}$$

where for $\gamma = S^2 + (n-1)T^2 - 2cST - (n-2)cT^2$, with $s_n > \gamma$ and $\theta = M^2 + (n-1)N^2 - 2bMN - (n-2)bN^2$, with $s_1 \geq \theta$ if $b = 1$, or $s_1 > \theta$ if $b > 1$,

$$\psi_n = \frac{s_n + \gamma + \sqrt{(s_n - \gamma)^2 + 4c[2ST + (n-2)T^2]\sum_{i=1}^{n-1}(s_i - s_n)}}{2},$$

$$\Psi_l = \frac{s_l + \theta + \sqrt{(s_l - \theta)^2 + 4b[2MN + (n-2)N^2]\sum_{i=1}^{l-1}(s_i - s_l)}}{2}.$$

- If $l = 1$, $\Psi_l = s_1$ and if $T = 0$, $\psi_n = s_n$, hence (6) yields

$$\sqrt{s_n} \leq \rho(A) \leq \sqrt{s_1}, \qquad \text{[X.D. Zhang, J.S. Li(2002)].}$$

Sharp bounds for $\rho(A)$ part V (new)

Let $A \geq 0$ with $r_i > 0$, $1 \leq i \leq n$ and let $w_i = \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{n} a_{jk} \sum_{p=1}^{n} a_{kp} \frac{r_p}{r_i}$ be the i-th average 4-row sum of A with $w_1 \geq \cdots \geq w_n$.

7 Then $\sqrt[3]{z_n} \leq \rho(A) \leq \min\left\{\sqrt[3]{Z_l} : l = 1, \ldots, n\right\}$, where

$$z_n = \frac{w_n + \beta_1 - c\beta_2 + \sqrt{(w_n - \beta_1 + c\beta_2)^2 + 4c\beta_2 \sum_{i=1}^{n-1}(w_i - w_n)}}{2},$$

$$Z_l = \frac{w_l + \alpha_1 - b\alpha_2 + \sqrt{(w_l - \alpha_1 + b\alpha_2)^2 + 4b\alpha_2 \sum_{i=1}^{l-1}(w_i - w_l)}}{2},$$

$$\begin{aligned}
\beta_1 &= S^3 + (n^2 - 3n + 2)T^3 + 3(n-1)ST^2, \\
\beta_2 &= (n^2 - 3n + 3)T^3 + 3(n-2)ST^2 + 3S^2T, \\
\alpha_1 &= M^3 + (n^2 - 3n + 2)N^3 + 3(n-1)MN^2, \\
\alpha_2 &= (n^2 - 3n + 3)N^3 + 3(n-2)MN^2 + 3M^2N.
\end{aligned}$$

- If $l = 1$ and $T = 0$, then $\sqrt[3]{w_n} \leq \rho(A) \leq \sqrt[3]{w_1}$.

Sharp bounds for $\rho(A)$ part VI (new)

Let $A \geq 0$ with $r_i > 0$, $1 \leq i \leq n$ and let the i-th average 5-row sum of A $v_i = \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{n} a_{jk} \sum_{p=1}^{n} a_{kp} \sum_{q=1}^{n} a_{pq} \frac{r_q}{r_i}$ with $v_1 \geq \cdots \geq v_n$.

8 Then $\sqrt[4]{\omega_n} \leq \rho(A) \leq \min\left\{\sqrt[4]{\Omega_l} : l = 1, \ldots, n\right\}$, where

$$\omega_n = \frac{v_n + \hat{\beta}_1 - c\hat{\beta}_2 + \sqrt{(v_n - \hat{\beta}_1 + c\hat{\beta}_2)^2 + 4c\hat{\beta}_2 \sum_{i=1}^{n-1}(v_i - v_n)}}{2},$$

$$\Omega_l = \frac{v_l + \hat{\alpha}_1 - b\hat{\alpha}_2 + \sqrt{(v_l - \hat{\alpha}_1 + b\hat{\alpha}_2)^2 + 4b\hat{\alpha}_2 \sum_{i=1}^{l-1}(v_i - v_l)}}{2},$$

$$\begin{aligned}
\hat{\beta}_1 &= S^4 + (n-1)(n^2-3n+3)T^4 + 4(n-1)(n-2)ST^3 + 6(n-1)S^2T^2, \\
\hat{\beta}_2 &= (n-1)(n^2-3n+2)T^4 + 4(n^2-3n+3)ST^3 + 6(n-2)S^2T^2 + 4S^3T, \\
\hat{\alpha}_1 &= M^4 + (n-1)(n^2-3n+3)N^4 + 4(n-1)(n-2)MN^3 + 6(n-1)M^2N^2, \\
\hat{\alpha}_2 &= (n-1)(n^2-3n+2)N^4 + 4(n^2-3n+3)MN^3 + 6(n-2)M^2N^2 + 4M^3N.
\end{aligned}$$

- If $l = 1$ and $T = 0$, then $\sqrt[4]{v_n} \leq \rho(A) \leq \sqrt[4]{v_1}$.

Example

Consider the 4×4 nonnegative matrix $A = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ with spectral radius $\rho(A) = 6.00000000000002$.

Table: Numerical comparison of all bounds for $\rho(A)$.

	lower bound	upper bound
Frobenius (1912)	5	7
Brauer, Gentry (1974)	5.4641016	6.5311288
Melman (2013)	5.5615528	6.6457513
Duan, Zhou (2013)	5.8284271	6.3722813
Xing, Zhou (2014)	5.9110419	6.1258376
Lin, Zhou (2017)	5.9379857	6.1116928
Our bounds (7th formula)	5.9513318	6.1040599
Our bounds (8th formula)	5.9592629	6.0919455

Example

Let the 5×5 nonnegative 2-tridiagonal Toeplitz $T_5^{(2)}(1,3,9) = \begin{bmatrix} 3 & 0 & 9 & 0 & 0 \\ 0 & 3 & 0 & 9 & 0 \\ 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$

with spectral radius $\rho(T_5^{(2)}) = 7.24264068$.

Table: Numerical comparison of all bounds for $\rho(T_5^{(2)})$

	lower bound	upper bound
Frobenius (1912)	4	13
Brauer, Gentry (1974)	4	12.48683298
Melman (2013)	6	12
Duan, Zhou (2013)	4	12.48683298
Xing, Zhou (2014)	6	11.39865155
Lin, Zhou (2017)	6	10.17349497
Our bounds (7th formula)	6	9.8488578
Our bounds (8th formula)	6	8.73885189

Application to Generalized Fibonacci matrices

Let $k = 1, 2, \ldots, m = 0, 1, \ldots$, and $c_1, c_2, \ldots, c_k \geq 0$. Then for every $n \geq k + m + 1$ the n-th term of the *generalized k, m-step Fibonacci sequence* $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$ is given by the recursive formula

$$f_n = c_1 f_{n-m-1} + c_2 f_{n-m-2} + \cdots + c_k f_{n-m-k}, \text{ with } f_1 = \ldots = f_{k+m} = 1.$$

[M. Adam, N. Assimakis and G. Tziallas (2015)].

For $c_1 = \ldots = c_k = 1$ the generalized k, m-step Fibonacci sequence $\left(f_n^{\{k,m\}}(1, 1, \ldots, 1)\right)_{n=1,2,\ldots}$ gives known sequences for various k, m:

1. For $k = 2$, $m = 0$, we have the well-known Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

2. For $k = 2$, $m = 1$, we have the Padovan sequence

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, \ldots$$

If $k \geq 2$ and $c_1 > 0$, $\left(f_n^{\{k,m\}}(c_1, c_2, \ldots, c_k)\right)_{n=1,2,\ldots}$ is associated to the $(k+m) \times (k+m)$ *generalized k, m-Fibonacci matrix*

$$R_{k,m}(c_1, \ldots, c_k) = \left[\begin{array}{cccc|c} 0_{1\times m} & c_1 & \cdots & c_{k-1} & c_k \\ \hline \multicolumn{4}{c|}{I_{(k+m-1)\times(k+m-1)}} & 0_{(k+m-1)\times 1} \end{array}\right].$$

If $m = 0$, then $R_{k,0}(c_1, \ldots, c_k)$ restricts to the *generalized k-Fibonacci matrix*

$$Q_k(c_1, \ldots, c_k) = \left[\begin{array}{ccc|c} c_1 & \cdots & c_{k-1} & c_k \\ \hline \multicolumn{3}{c|}{I_{(k-1)\times(k-1)}} & 0_{(k-1)\times 1} \end{array}\right].$$

- $Q_k(c_1, \ldots, c_k)$ is nonsingular if and only if $c_k \neq 0$, since

$$\det(Q_k(c_1, \ldots, c_k)) = (-1)^{k+1} c_k.$$

- $Q_k(c_1, \ldots, c_k)$ is irreducible, since

$$(I_k + Q_k(c_1, \ldots, c_k))^{k-1} > 0, \quad k \geq 2.$$

- $Q_k(c_1, \ldots, c_k)$ is primitive, since

$$Q_k(c_1, \ldots, c_k)^{k^2-2k+2} > 0.$$

- Due to the irreducibility property, the spectral radius $\rho(Q_k(c_1,\ldots,c_k))$ is a positive and simple (without multiplicity) eigenvalue of $Q_k(c_1,\ldots,c_k)$.

- If all the distinct eigenvalues of the nonsingular matrix $Q_k(c_1,\ldots,c_k)$ $(c_k > 0)$ are denoted by $\lambda_1,\ldots,\lambda_s,\rho(Q_k(c_1,\ldots,c_k))$, with $s \leq k-1$, then the following inequality holds

$$0 < |\lambda_j| \leq \rho(Q_k(c_1,\ldots,c_k)),\ j = 1,\ldots,s.$$

- Due to the primitivity property, $\rho(Q_k(c_1,\ldots,c_k)) > 0$ is the unique eigenvalue of $Q_k(c_1,\ldots,c_k)$ of maximum modulus.

- This yields that the remaining eigenvalues $\lambda_1,\ldots,\lambda_s$ lie in the interior of the spectral disk, namely,

$$|\lambda_j| < \rho(Q_k(c_1,\ldots,c_k)),\ j = 1,\ldots,s.$$

Suppose $c_i > 1$ for all $i = 1, \dots, k$, and let $C = \sum_{i=1}^{k} c_i$, $c_p = \max_{2\leq i\leq k} c_i$. Then we apply the aforementioned bounds to $Q_k := Q_k(c_1, \dots, c_k)$:

1. Frobenius (1912): $1 \leq \rho(Q_k) \leq C$.
2. Brauer, Gentry (1974): $1 \leq \rho(Q_k) \leq \frac{c_1}{2} + \sqrt{\left(\frac{c_1}{2}\right)^2 + C - c_1}$.
3. Melman (2013): Let
   $\tau_1 = \max\{c_1 + \sqrt{c_1^2 + 4(C - c_1)}, 1 + c_1 + \sqrt{(1 - c_1)^2 + 4(C - c_1)}\}$
   and
   $\tau_2 = \min\{c_1 + \sqrt{c_1^2 + 4(C - c_1)}, 1 + c_1 + \sqrt{(1 - c_1)^2 + 4(C - c_1)}\}$,
   $$\min_{1\leq i\leq k}\left\{\frac{\tau_1}{2}, \frac{C - c_i}{2} + \sqrt{\left(\frac{C - c_i}{2}\right)^2 + c_i}\right\} \leq \rho(Q_k) \leq \frac{\tau_2}{2}.$$
4. Duan, Zhou (2013):
   $$1 \leq \rho(Q_k) \leq \frac{1 + c_1 - c_p}{2} + \sqrt{\left(\frac{1 - c_1 + c_p}{2}\right)^2 + c_p(C - 1)}.$$

5 Xing, Zhou (2014):

$$1 \le \rho(Q_k) \le \min_{1\le l\le k} X_l,$$

where

$$\begin{aligned} X_1 &= m_1 = C, \\ X_l &= \frac{m_l + c_1 - Cc_p}{2} + \sqrt{\left(\frac{m_l - c_1 + Cc_p}{2}\right)^2 + Cc_p\left(\sum_{i=1}^{l} m_i - lm_l\right)}, \end{aligned}$$

for $l = 2, 3$ and $X_4 = \cdots = X_k = X_3$ with

$$\begin{aligned} m_2 &= c_1\left(1 - \frac{1}{C}\right) + 1, \\ m_3 &= m_4 = \cdots = m_k = 1. \end{aligned}$$

6 Lin, Zhou (2017): If $c_1^2 + c_1 + c_2 < C$, then

$$1 \leq \rho(Q_k) \leq \min_{1 \leq l \leq k} \sqrt{\Psi_l}, \quad \text{where}$$

$$\begin{aligned}
\Psi_1 &= s_1 = C + c_1(C-1), \\
\Psi_l &= \frac{s_l + \alpha}{2} + \sqrt{\left(\frac{s_l - \alpha}{2}\right)^2 + \frac{C\alpha - C(c_1 - c_p)^2}{1 - C}\left(\sum_{i=1}^{l} s_i - l s_l\right)},
\end{aligned}$$

for $l = 2, 3, 4$ and $\Phi_5 = \cdots = \Psi_k = \Psi_4$ with

$$\begin{aligned}
s_2 &= C, \\
s_3 &= (c_1^2 + c_1 + c_2)\left(1 - \frac{1}{C}\right) + 1, \\
s_4 &= s_5 = \cdots = s_k = 1, \\
\alpha &= (1 - C)\beta + (c_1 - c_p)^2, \\
\beta &= 2c_1 c_p + (k-2)c_p^2.
\end{aligned}$$

7 Our bounds:

$$\sqrt[3]{C} \le \rho(Q_k) \le \min_{1\le l\le k} \sqrt[3]{Z_l}, \quad \text{where}$$

$$\begin{aligned}
Z_1 &= w_1 = (c_1^2 + c_1 + c_2)(C-1) + C,\\
Z_l &= \frac{w_l+\alpha}{2} + \sqrt{\left(\frac{w_l-\alpha}{2}\right)^2 + \frac{C\alpha - C(c_1-c_p)^3}{1-C}\left(\sum_{i=1}^{l} w_i - l w_l\right)},
\end{aligned}$$

for $2 \le l \le 5$ and $Z_6 = \cdots = Z_k = Z_5$ with

$$\begin{aligned}
w_2 &= [c_1(c_1^2+c_1+c_2) + c_1c_2 + c_1 + c_2 + c_3]\left(1 - \frac{1}{C}\right) + 1,\\
w_3 &= c_1(C-1) + C,\\
w_4 &= C,\\
w_5 &= w_6 = \cdots = w_k = 1,\\
\alpha &= (1-C)\beta + (c_1 - c_p)^3,\\
\beta &= (k^2 - 3k + 3)c_p^3 + 3(k-2)c_1c_p^2 + 3c_1^2c_p.
\end{aligned}$$

8 Our bounds:

$$\sqrt[4]{c_1(C-1)+C} \le \rho(Q_k) \le \min_{1\le l\le k} \sqrt[4]{\Omega_l}, \quad \text{where}$$

$$\begin{aligned}
\Omega_1 &= v_1 = [c_1(c_1^2+c_1+c_2)+c_1c_2+c_1+c_2+c_3](C-1)+C, \\
\Omega_l &= \frac{v_l+\alpha}{2} + \sqrt{\left(\frac{v_l-\alpha}{2}\right)^2 + \frac{C\alpha - C(c_1-c_p)^4}{1-C}\left(\sum_{i=1}^{l} v_i - l v_l\right)},
\end{aligned}$$

for $2 \le l \le 6$ and $\Omega_7 = \cdots = \Omega_k = \Omega_6$ with

$$\begin{aligned}
v_2 &= [(c_1^2+c_2+1)(c_1^2+c_1+c_2)+c_1c_2(c_1+1)+2c_1c_3+c_3+c_4]\left(1-\frac{1}{C}\right)+1 \\
v_3 &= (c_1^2+c_1+c_2)(C-1)+C, \\
v_4 &= c_1(C-1)+C, \\
v_5 &= C, \\
v_6 &= v_7 = \cdots = v_k = 1, \\
\alpha &= (1-C)\beta + (c_1-c_p)^4, \\
\beta &= (k-1)(k^2-3k+2)c_p^4 + 4(k^2-3k+3)c_1c_p^3 + 6(k-2)c_1^2c_p^2 + 4c_1^3c_p.
\end{aligned}$$

Example

Let the generalized 4, 0-step Fibonacci matrix $Q_4 = R_{4,0} = \begin{bmatrix} 3 & 12 & 2 & 8 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ with spectral radius $\rho(Q_4) = 5.360252450371726$.

Table: Numerical comparison of all bounds for $\rho(Q_4)$

	lower bound	upper bound
Frobenius (1912)	1	25
Brauer, Gentry (1974)	1	6.4244289
Melman (2013)	3	13.8654599
Duan, Zhou (2013)	1	13.6918060
Xing, Zhou (2014)	1	23.6404512
Lin, Zhou (2017)	1	9.8488578
Our bounds (7th formula)	2.9240177	8.4390097
Our bounds (8th formula)	3.1382889	7.4161984

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