The computation of the greatest common divisor of three Bernstein basis polynomials

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Introduction

The computation of the greatest common divisor (GCD) of two polynomials has been studied in detail.

- The Bernstein basis is used extensively in geometric modelling because of its elegant geometric properties and superior numerical properties with resect to the power basis
- The GCD of two or three bivariate polynomials in the Bernstein basis is required for the calculation of the points of intersection of surfaces, which is one of the most important problems in geometric modelling

The computation of the GCD of bivariate Bernstein basis polynomials is a difficult problem and it is therefore appropriate to consider the computation of the GCD of three univariate Bernstein basis polynomials, before the more difficult bivariate polynomial GCD problem is addressed.

The formulation of the problem

Consider three Bernstein basis polynomials $f(y), g(y)$ and $h(y)$ of degrees m, n and p, respectively,

$$f(y) = \sum_{i=0}^{m} a_i \binom{m}{i} (1-y)^{m-i} y^i$$

$$g(y) = \sum_{i=0}^{n} b_i \binom{n}{i} (1-y)^{n-i} y^i$$

$$h(y) = \sum_{i=0}^{p} c_i \binom{p}{i} (1-y)^{p-i} y^i$$

and let the degree of their GCD be t. There therefore exist polynomials $u_k(y), v_k(y)$ and $w_k(y)$ of degrees $m-k, n-k$ and $p-k$ respectively, such that

$$d_k(y) = \frac{f(y)}{u_k(y)} = \frac{g(y)}{v_k(y)} = \frac{h(y)}{w_k(y)}, \qquad k = 1, \ldots, t,$$

$$d_t(y) = \mathrm{GCD}(f, g, h)$$

It follows that the polynomials $d_k(y)$ are common divisors of $f(y), g(y)$ and $h(y)$ if the equations

$$f(y)v_k(y) - g(y)u_k(y) = 0$$
$$f(y)w_k(y) - h(y)u_k(y) = 0$$
$$g(y)w_k(y) - h(y)v_k(y) = 0$$

are satisfied for $k = 1, \ldots, t$.

- Any two of these equations defines the third equation. For example the combination of

  $$f(y)v_k(y) - g(y)u_k(y) = 0 \quad \text{and} \quad f(y)w_k(y) - h(y)u_k(y) = 0$$

  implies the third equation

  $$g(y)w_k(y) - h(y)v_k(y)$$

- This raises two options for GCD computations:
  - **Variant 1** Consider two pairs of equations
  - **Variant 2** Consider all three equations

- **Variant 1** Two pairs of equations:

Group 1:	$f(y) = u(y)d(y),$	$g(y) = v(y)d(y)$
Group 2:	$f(y) = u(y)d(y),$	$h(y) = w(y)d(y)$
Group 3:	$g(y) = v(y)d(y),$	$h(y) = w(y)d(y)$

Which pairs of equations should be considered?

$$((f, g), (f, h)), \qquad ((f, g), (g, h)) \qquad \text{and} \qquad ((f, h), (g, h))$$

which correspond to Groups 1 and 2, Groups 1 and 3, and Groups 2 and 3, respectively.
Although the GCD of $f(y), g(y)$ and $h(y)$ is independent of the pairs of Groups that are considered, the computational answers may differ.

- **Variant 2** Three equations: The problems of Variant 1 do not exist, but the matrices in this variant are larger than the matrices in Variant 1

The formulation of the problem

The Sylvester resultant matrix and its subresultant matrices are used for the GCD computations for Variants 1 and 2.

Variant 1

$$\bar{S}_k(f,g,h) = \begin{bmatrix} C_{n-k}(f) & & C_{m-k}(g) \\ & C_{p-k}(f) & C_{m-k}(h) \end{bmatrix}$$

order: $(2m+n+p-2k+2) \times (m+n+p-3k+3)$

$$\bar{S}_k(g,f,h) = \begin{bmatrix} C_{m-k}(g) & & C_{n-k}(f) \\ & C_{p-k}(g) & C_{n-k}(h) \end{bmatrix}$$

order: $(m+2n+p-2k+2) \times (m+n+p-3k+3)$

$$\bar{S}_k(h,g,f) = \begin{bmatrix} C_{n-k}(h) & & C_{p-k}(g) \\ & C_{m-k}(h) & C_{p-k}(f) \end{bmatrix}$$

order: $(m+n+2p-2k+2) \times (m+n+p-3k+3)$

for $k = 1, \ldots, \min(m,n,p)$.

Variant 2

$$\tilde{S}_k(f,g,h) = \tilde{D}_k^{-1}\tilde{T}_k(f,g,h)\tilde{Q}_k, \qquad k = 1,\dots,\min(m,n,p)$$

is of order $(2(m+n+p)-3k+3)\times(m+n+p-3k+3)$.

1. The matrices $\bar{D}_k^{-1}$ and $\tilde{D}_k^{-1}$ contain terms of the form

$$\frac{1}{\binom{m+n-k}{i_1}}, \qquad \frac{1}{\binom{m+p-k}{i_2}} \qquad \text{and} \qquad \frac{1}{\binom{n+p-k}{i_3}}$$

2. The matrices $\bar{T}_k^{-1}$ and $\tilde{T}_k^{-1}$ contain the coefficients of the polynomials and terms of the form

$$\binom{m}{j_1}, \qquad \binom{n}{j_2} \qquad \text{and} \qquad \binom{p}{j_3}$$

3. The matrices $\bar{Q}_k^{-1}$ and $\tilde{Q}_k^{-1}$ contain terms of the form

$$\binom{m-k}{l_1}, \qquad \binom{n-k}{l_2} \qquad \text{and} \qquad \binom{p-k}{l_3}$$

The Sylvester resultant matrix

The forms of the entries of the Sylvester matrices for Variants 1 and 2, $\bar{S}_k(f,g,h)$ and $\tilde{S}_k(f,g,h)$ respectively, are very similar.

Theorem

The degree t of the GCD of $f(y), g(y)$ and $h(y)$ is equal to the largest integer k such that $\bar{S}_k(f,g,h)$ and $\tilde{S}_k(f,g,h)$ are singular,

$$\begin{aligned} \text{rank } \bar{S}_k(f,g,h) &< m+n+p-3k+3, & k &= 1,\ldots,t, \\ \text{rank } \bar{S}_k(f,g,h) &= m+n+p-3k+3, & k &= t+1,\ldots,q, \end{aligned}$$

$$\begin{aligned} \text{rank } \tilde{S}_k(f,g,h) &< m+n+p-3k+3, & k &= 1,\ldots,t, \\ \text{rank } \tilde{S}_k(f,g,h) &= m+n+p-3k+3, & k &= t+1,\ldots,q, \end{aligned}$$

where $q = \min(m,n,p)$.

The coefficients of the GCD of $f(y), g(y)$ and $h(y)$ lie in the nullspace of $\bar{S}_t(f,g,h)$ and $\tilde{S}_t(f,g,h)$. □

The matrices $\bar{D}_k^{-1}, \tilde{D}_k^{-1}, \bar{Q}_k$ and $\tilde{Q}_k$ are non-singular, and thus

Variant 1:
$$\text{rank } \bar{T}_k = \text{rank } \bar{T}_k \bar{Q}_k = \text{rank } \bar{D}_k^{-1} \bar{T}_k = \text{rank } \bar{D}_k^{-1} \bar{T}_k \bar{Q}_k$$

Variant 2:
$$\text{rank } \tilde{T}_k = \text{rank } \tilde{T}_k \tilde{Q}_k = \text{rank } \tilde{D}_k^{-1} \tilde{T}_k = \text{rank } \tilde{D}_k^{-1} \tilde{T}_k \tilde{Q}_k$$

The degree of the GCD of $f(y), g(y)$ and $h(y)$ can be calculated from:

Variant 1:
$$\bar{T}_k(f,g,h),\ \bar{D}_k^{-1} \bar{T}_k(f,g,h),\ \bar{T}_k(f,g,h)\bar{Q}_k,\ \bar{D}_k^{-1} \bar{T}_k(f,g,h)\bar{Q}_k$$

Variant 2:
$$\tilde{T}_k(f,g,h),\ \tilde{D}_k^{-1} \tilde{T}_k(f,g,h),\ \tilde{T}_k(f,g,h)\tilde{Q}_k,\ \tilde{D}_k^{-1} \tilde{T}_k(f,g,h)\tilde{Q}_k$$

- The entries in these matrices may span many orders of magnitude because of the combinatorial terms, and this may lead to incorrect results

Example 1 Let the degrees of the polynomials $f(y), g(y)$ and $h(y)$ be $m = 5, n = 15$ and $p = 7$, respectively, and let the coefficients of the polynomials be one because it is desired to consider the effects of the combinatorial terms in the different forms of Variants 1 and 2.

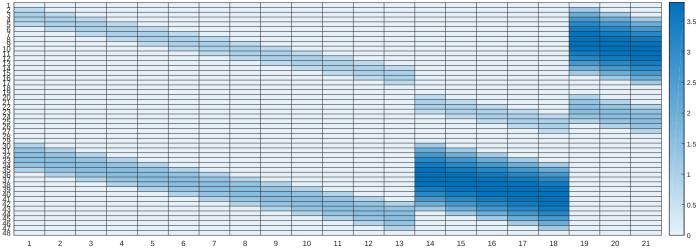

Figure: Heat map of the combinatorial terms in $\bar{T}_3(f, g, h)$ for Example 1.

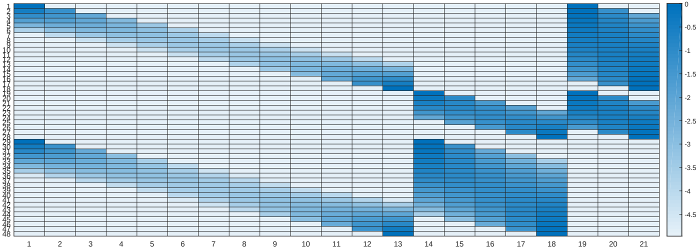

Figure: Heat map of the combinatorial terms in $\bar{D}_3^{-1}\bar{T}_3(f,g,h)$ for Example 1.

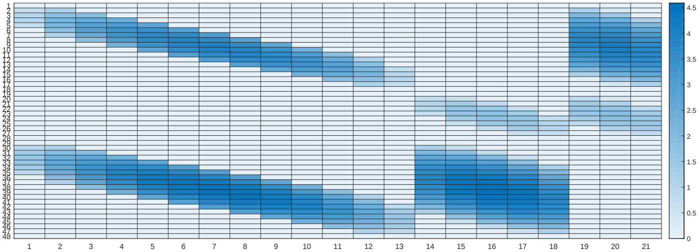

Figure: Heat map of the combinatorial terms in $\bar{T}_3(f,g,h)\bar{Q}_3$ for Example 1.

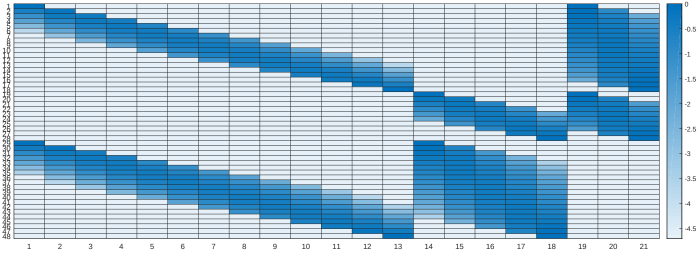

Figure: Heat map of the combinatorial terms in $\bar{D}_3^{-1}\bar{T}_3(f,g,h)\bar{Q}_3$ for Example 1.

The ratio of the maximum entry of a matrix to its minimum entry is a minimum for the matrix $\bar{D}_3^{-1}\bar{T}_3(f,g,h)\bar{Q}_3$, which is expected from a comparison of the combinatorial terms:

$$\left\{ \binom{m}{i}, \quad \frac{\binom{m}{i}}{\binom{m+n-k}{j}}, \quad \binom{m}{i}\binom{n-k}{j-i}, \quad \frac{\binom{m}{i}\binom{n-k}{j-i}}{\binom{m+n-k}{j}} \right\}$$

□

Example 2 Consider the power basis polynomials $f(y)$, $g(y)$ and $h(y)$ of degrees $m = 17$, $n = 13$ and $p = 14$. The power basis is considered in order to eliminate adverse scaling that may arise from the combinatorial terms in the Bernstein basis functions.

$$f(y) = (y - 4.65)^2(y - 1.50)^2(y - 1.26)^4(y - 1.10)^3(y - 1)^4(y + 3)^2$$
$$g(y) = (y - 4.99)^3(y - 4.65)^2(y - 1.26)^4(y - 1.10)^3(y - 2)$$
$$h(y) = (y - 4.65)^2(y - 3.20)^2(y - 1.26)^4(y - 0.71)^4(y + 1.75)^2$$

It follows that

$$\begin{aligned} d_t(y) &= \mathrm{GCD}\,(f, g, h) = \mathrm{GCD}\,(f, h) = \mathrm{GCD}\,(g, h) \\ &= (y - 1.26)^4(y - 4.65)^2 \\ d_{f,g}(y) &= \mathrm{GCD}\,(f, g) \\ &= (y - 1.26)^4(y - 4.65)^2(y - 1.10)^3 \end{aligned}$$

and thus $t = \deg \mathrm{GCD}\,(f, g, h) = 6$ and $t_{f,g} = \deg \mathrm{GCD}\,(f, g) = 9$. The polynomials were normalised to unit 2-norm, and then scaled

$$\hat{f}(y) = 10^5 f(y), \qquad \hat{g}(y) = 10^5 g(y), \qquad \hat{h}(y) = 10^{-5} h(y)$$

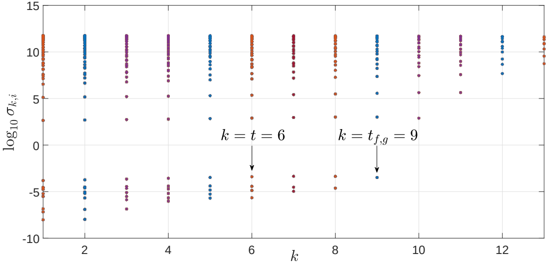

Figure: The singular values $\sigma_{k,i}$ of $\tilde{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 2.

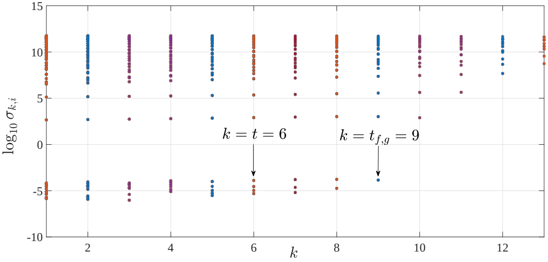

Figure: The singular values $\sigma_{k,i}$ of $\bar{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 2.

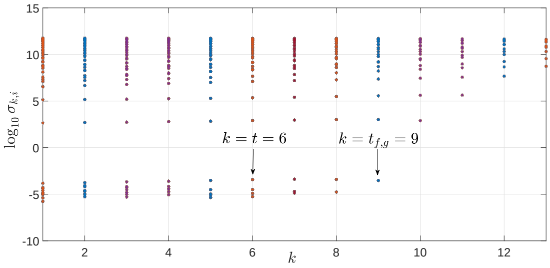

Figure: The singular values $\sigma_{k,i}$ of $\bar{S}_k(\hat{g}, \hat{f}, \hat{h})$ for Example 2.

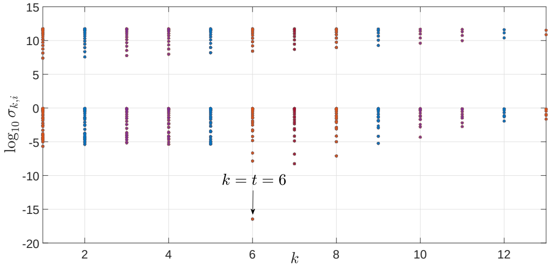

Figure: The singular values $\sigma_{k,i}$ of $\bar{S}_k(\hat{h}, \hat{g}, \hat{f})$ for Example 2.

Preprocessing operations

Variants 1 and 2 may lead to incorrect results, even in the absence of noise, but considerably improved results are obtained when $f(y), g(y)$ and $h(y)$ are preprocessed by three operations:

Operation 1 Normalisation of the non-zero entries in each Sylvester matrix and subresultant matrix.

Operation 2 Two of the polynomials, $f(y)$ and $h(y)$, are multiplied by constants λ_k and ρ_k respectively,

$$\mathrm{GCD}\,(f, g, h) \sim \mathrm{GCD}\,(\lambda_k f, g, \rho_k h)$$

where $\sim$ denotes equivalence to within an arbitrary non-zero constant. The subscript k is included in the constants λ_k and ρ_k because their optimal values must be calculated for each subresultant matrix.

Operation 3 Perform a transformation from y to a new independent variable w, which are related by the constant θ_k that is calculated for each subresultant matrix

$$y = \theta_k w$$

Example 3 Consider again the power basis polynomials $f(y)$, $g(y)$ and $h(y)$ of degrees $m = 17$, $n = 13$ and $p = 14$, that are scaled by $10^5, 10^5$ and 10^{-5}, respectively, and then preprocessed.

$$f(y) = (y - 4.65)^2(y - 1.50)^2(y - 1.26)^4(y - 1.10)^3(y - 1)^4(y + 3)^2$$
$$g(y) = (y - 4.99)^3(y - 4.65)^2(y - 1.26)^4(y - 1.10)^3(y - 2)$$
$$h(y) = (y - 4.65)^2(y - 3.20)^2(y - 1.26)^4(y - 0.71)^4(y + 1.75)^2$$

It follows that $\deg \text{GCD}\,(f, g, h) = 6$ and $\deg \text{GCD}\,(f, g) = 9$.

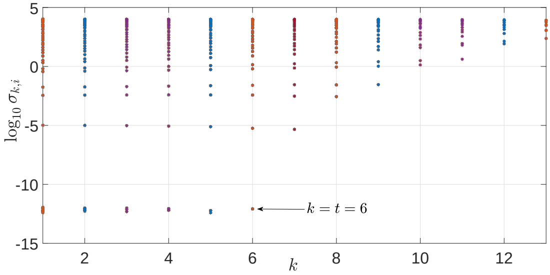

Figure: The singular values of $\tilde{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 3.

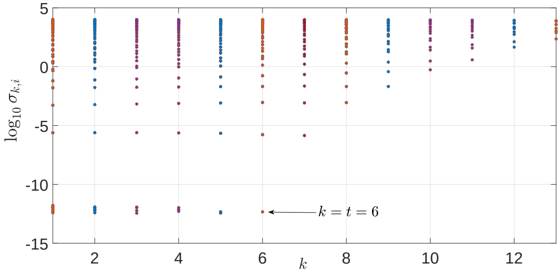

Figure: The singular values of $\bar{S}_k(\hat{f}, \hat{g}, \hat{h})$ for Example 3.

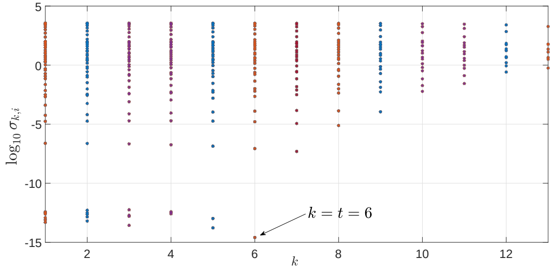

Figure: The singular values of $\bar{S}_k(\hat{g}, \hat{f}, \hat{h})$ for Example 3.

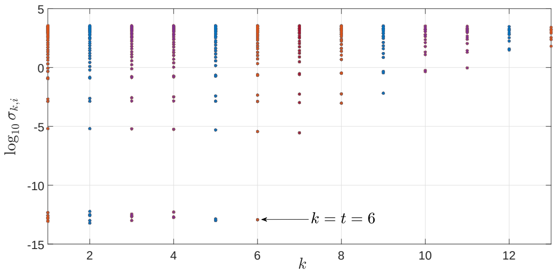

Figure: The singular values of $\bar{S}_k(\hat{h}, \hat{g}, \hat{f})$ for Example 3.

Preprocessing the polynomials yields a significant improvement in the results, even though they are badly scaled. □

Example 4 Consider the Bernstein forms of the exact polynomials $\hat{f}(y)$, $\hat{g}(y)$ and $\hat{h}(y)$, of degrees $12, 36$ and 15, whose factored forms are

$$\begin{aligned}\hat{f}(y) &= (y-0.5654654561)^5(y-0.21657894)(y-0.01564897)^2 \times \\ &\quad (y+0.2468796514)^3(y+0.7879734) \\ \hat{g}(y) &= (y-0.99851354877)^7(y-0.75292)^{20}(y-0.5654654561)^5 \times \\ &\quad (y-0.21657894)(y+0.2468796514)^3 \\ \hat{h}(y) &= (y-0.5654654561)^5(y-0.21657894)(y+0.2468796514)^3 \times \\ &\quad (y+0.778912324654)^4(y+1.75)^2\end{aligned}$$

whose GCD $\hat{d}_t(y)$ is of degree $t=9$

$$\hat{d}_t(y) = (y-0.5654654561)^5(y-0.21657894)(y+0.2468796514)^3$$

The polynomial $\hat{g}(y)$ has four roots in the unit interval, and one of these roots is of multiplicity 20.

The polynomials were processed and noise was added, thereby forming the polynomials $f(y), g(y)$ and $h(y)$, such that the upper bound of the componentwise relative error is a uniformly distributed random variable in the interval $\left[10^{-7}, 10^{-4}\right]$. $\square$

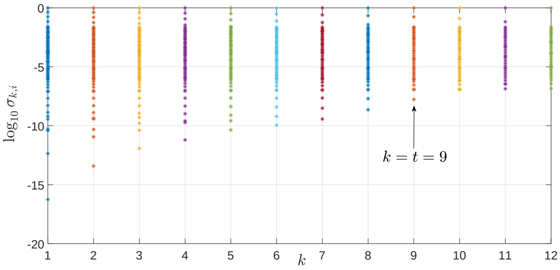

Figure: The singular values of $\bar{S}_k(f, g, h)$, before preprocessing, for Example 4.

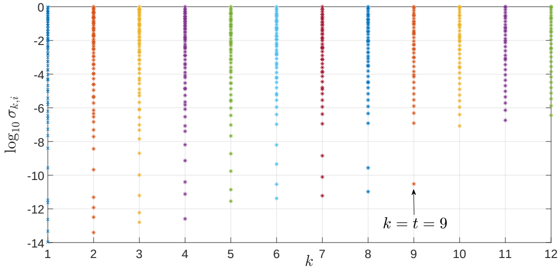

Figure: The singular values of $\bar{S}_k(f, g, h)$, after preprocessing, for Example 4.

Summary

- There are two formulations for the problem of the computation of the GCD of three polynomials. The first formulation requires that two pairs of polynomials be considered, and the second formulation requires that three pairs of polynomials be considered.
- Both formulations can be implemented using the Sylvester resultant matrix and its subresultant matrices. The dimensions of the matrices differ between the formulations, but the matrices share the same structure.
- The combinatorial terms in the Bernstein basis functions may lead to computational errors because of the large range of magnitudes in the entries of the matrices. The adverse effects of this large range of magnitudes can be significantly reduced, in the two formulations, by processing the polynomials before computations are performed on their Sylvester matrix and subresultant matrices.