Review of the convergence of some Krylov subspace methods for solving linear systems of equations with one or several right hand sides

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We consider the linear systems of equations with multiple right-hand sides

\[ AX = F, \quad A \in \mathbb{C}^{n \times n}, \quad F \in \mathbb{C}^{n \times s}, \quad X \in \mathbb{C}^{n \times s}, \quad 1 \leq s \ll n. \]  

(1)

Which is equivalent to the \(s\)-linear systems of equations

\[ Ax^{(i)} = f^{(i)}, \quad \text{for} \quad i = 1, \ldots, s, \]  

(2)

To solve problem (1), several block methods have been developed.

- Standard GMRES (Classical GMRES)
- Block GMRES
- Global GMRES

Reference

Consider the linear system of equations

\[ A x^{(i)} = f^{(i)} \]

For the GMRES method, the iterates \( \{x_k^{(i)}\} \) are defined by the following conditions

**Standard-GMRES**

\[
x_k^{(i)} - x_0^{(i)} \in K_k(A, r_0^{(i)}), \]
\[
(A^j r_0^{(i)}, r_k^{(i)}) = 0 \quad \text{for} \quad j = 1, \ldots, k,
\]

where \( r_0^{(i)} = f^{(i)} - A x_0^{(i)} \)

**reference**

definition: Matrix Krylov subspace

Let

\[ \mathbf{K}^G_k(A, U) = \text{span}\{U, AU, \ldots, A^{k-1}U\} \subset \mathbb{C}^{n \times s} \]

denote the matrix Krylov subspace spanned by the matrices \( U, AU, \ldots, A^{k-1}U \), where \( U \) is an \( n \times s \) matrix. Note that \( Z \in \mathbf{K}^G_k(A, U) \) implies that

\[ Z = \sum_{j=1}^{k} \alpha_j A^{j-1}U, \quad \alpha_j \in \mathbb{C}, \quad j = 1, \ldots, k. \]

Reference

Global-GMRES

The global GMRES method constructs, at step $k$, the approximation $X_k$ satisfying the following two relations

$$X_k - X_0 \in K_k^G(A, R_0) \text{ and } \langle A^j R_0, R_k \rangle_F = 0, \quad \text{for } j = 1, \ldots, k,$$

where we define the inner product $\langle Y, Z \rangle_F = \text{trace}(Y^H Z)$ (where $Y^H$ denotes the conjugate transpose of $Y$). The associated norm is the Frobenius norm $\| . \|_F$.

We can also use the following inner product

$$\langle Y, Z \rangle = \sum_{1 \leq i, j \leq s} Y_{i}^H Z_{j}$$

Convergence

If $d$ is the degree of the minimal polynomial of $A$ with respect to $R_0$, then

$$R_d = 0.$$
Global-GMRES

The residual $R_k = F - AX_k$ satisfies the minimization property

$$\|R_k\|_F = \min_{Z \in K_k^G(A, R_0)} \|R_0 - AZ\|_F.$$  (3)
Definition of BGMRES method

\( \mathbb{K}_k(A, R_0) \) denote the matrix Krylov subspace defined as

\[
\mathbb{K}_k(A, R_0) = \text{blockspan}\{R_0, AR_0, \ldots, A^{k-1}R_0\}
\]

\[
= \left\{ \sum_{i=1}^{k} A^k R_0 \Omega_i / \Omega_i \in \mathbb{C}^{s \times s} \right\}.
\]

Let

\[
\mathcal{B}_k(A, R_0) = \sum_{i=1}^{s} \mathbb{K}_k(A, r_0^{(i)}).
\]

For \( i = 1, \ldots, s \) we have \( x_k^{(i)} \in x_0^{(i)} + \mathcal{B}_k(A, R_0) \).

reference

For nonsymmetric problems, the BGMRES [Vital 1990] generate an approximate solution $X_k$ over the matrix krylov subspace $\mathbb{K}_k(A, R_0)$ with orthogonality condition on the residual $R_k$.

**Block GMRES**

\[
\begin{align*}
X_k - X_0 &\in \mathbb{K}_k(A, R_0) \\
R_k &= F - AX_k \perp A\mathbb{K}_k(A, R_0),
\end{align*}
\]

**Reference**


Block GMRES Algorithm

The definition of BGMRES is equivalent to

\[ X_k = X_0 + Z_k \]

\[ \| R_0 - AZ_k \|_F = \min_{Z \in \mathbb{K}_k(A,R_0)} \| R_0 - AZ \|_F. \]  

(6)

Thus BGMRES method proceeds as follows

**ALGORITHM (BGMRES)**

1. Choose \( X_0 \in \mathbb{C}^{N \times s} \) and compute \( R_0 = F - AX_0 \).
2. \( R_0 = V_1 H_{1,0} \) (The QR factorisation of \( R_0 \));
3. For \( j=1, \ldots, k \), do
   construct \( V_j \) and \( \widehat{H}_j \) by block Arnoldi.
4. Solve the least squares problem:
   \[ Y_k = \arg \min_{Y \in \mathbb{C}^{k \times s}} \| R_0 - AV_k Y \|_F, \]
5. The approximate solution is \( X_k = X_0 + V_k Y_k \).
Let $U$ be an $N \times s$ matrix. An Arnoldi-type algorithm constructs a basis 
{$\{V_1^\bullet, \ldots, V_k^\bullet\}$} of $K_k^B(A, U)$, which satisfies an orthogonal property. Moreover the block matrix $V_k^\bullet = [V_1^\bullet, \ldots, V_k^\bullet]$ is such that the matrix $H_k^\bullet = V_k^H A V_k^\bullet$ is an upper block Hessenberg.

We examine three possibly choices of orthogonality. Let $\Phi^\bullet$ be the map : $\mathbb{C}^{n\times s} \times \in \mathbb{C}^{n\times s} \rightarrow \mathbb{C}^{s\times s}$ defined for $\bullet \in \{B, S, G\}$ by

$$
\begin{align*}
\Phi^B(X, Y) &= X^H Y, \\
\Phi^S(X, Y) &= \text{the diagonal of the matrix } (X^H Y), \\
\Phi^G(X, Y) &= \text{trace}(X^H Y)I_s = \langle X, Y \rangle_F I_s,
\end{align*}
$$

$\forall X \in \mathbb{C}^{n\times s}$ and $\forall Y \in \mathbb{C}^{n\times s}$.

If $\Phi^B(X, Y) = X^H Y = 0$, then the block-vectors $X, Y$ are called block-orthogonal by Gutknecht. Similarly $X$ is called block-normalized if $X^H X = I_s$. Of course the vector space of $\mathbb{C}^{n\times s}$ of block vectors is an Euclidean space, which is a finite-dimensional inner product space with the inner product : $\langle X, Y \rangle_F = \text{trace}(X^H Y)$. If $\langle X, Y \rangle_F = 0$, then $X$ and $Y$ are called F-orthogonal. If $\Phi^S(X, Y) = \text{diag}(X^H Y) = 0$, then $X$ and $Y$ can be called diagonally orthogonal.
The Block Arnoldi-type algorithm

1. Let $U$ be an $n \times s$ matrix.

2. Compute $V_1^* \in \mathbb{C}^{n \times s}$ by determining the factorization of $U$:
   
   $$U = V_1^* H_{1,0}^*, \quad H_{1,0}^* \in \mathbb{C}^{s \times s},$$

   such that $H_{1,0}^* = \Phi^*(V_1^*, U)$ and $\Phi^*(V_1^*, V_1^*) = I_s$.

3. for $i=1, \ldots, k$ do
   
   ▶ Compute $W = AV_i^*$.

   ▶ for $j=1, \ldots, i$ do
     
     1. $H_{j,i}^* = \Phi^*(V_j^*, W)$
     
     2. $W = W - V_j^* H_{j,i}^*$

   ▶ End

   ▶ Compute $H_{i+1,i}^*$ by determining the decomposition of $W$:

   $$W = V_{i+1}^* H_{i+1,i}^*,$$

   such that $H_{i+1,i}^* = \Phi^*(V_{i+1}^*, W)$ and $\Phi^*(V_{i+1}^*, V_{i+1}^*) = I_s$.

4. End
For $\Phi^\bullet(X,Y) = \Phi^B(X,Y) = X^H Y$, the preceding algorithm reduces to Block Arnoldi, which builds an orthonormal basis $\{V_1^B, \ldots, V_k^B\}$ such that the block matrix $V_k^B = [V_1^B, \ldots, V_k^B]$ satisfies $(V_k^B)^H V_k^B = I_{ks}$.

It is well known that

$$A V_k^B = V_k^B H_k^B + V_{k+1}^B H_{k+1,k} E_k^T,$$

where $E_k^T = [0_s, \ldots, O_s, I_s] \in \mathbb{R}^{s \times ms}$.

$$\tilde{H}_k = \begin{pmatrix} H_{1,1} & H_{1,2} & \ldots & H_{1,k-1} & H_{1,k} \\ H_{2,1} & H_{2,2} & \ldots & H_{2,k-1} & H_{2,k} \\ H_{3,1} & H_{3,2} & \ldots & H_{3,k-1} & H_{3,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{k,k-1} & H_{k,k} & \ldots & H_{k,k} & H_{k+1,k} \end{pmatrix}$$
ALGORITHM

The Block Arnoldi algorithm

1. Let $U$ be an $N \times s$ matrix.

2. Compute the $N \times s$ $V_1 \in \mathbb{R}^{N \times s}$ by finding the QR factorization of $U$:
   
   $U = V_1 R, R \in \mathbb{R}^{s \times s}$

3. for $i = 1, \ldots, k$ do
   
   ▶ Compute $W = AV_i$.
   
   ▶ for $j = 1, \ldots, i$ do
     
     1. $H_{j,i} = V_j^\top W$
     
     2. $W = W - V_j H_{j,i}$
   
   ▶ End

   ▶ Compute $H_{i+1,i}$ by finding the QR decomposition of $W$:
     
     $W = V_{i+1} H_{i+1,i}$

4. End
The Global-Arnoldi algorithm

1. Set $V_1 = V / \| V \|_F$.
2. For $j = 1, \ldots, k$. do
   
   $\tilde{V} = AV_j$,
   
   for $i = 1, \ldots, j$. do
   
   $h_{i,j} = \langle V_i, \tilde{V} \rangle_F$,
   
   $\tilde{V} = \tilde{V} - h_{i,j} V_i$,
   
   endfor
   
   $h_{j+1,j} = \| \tilde{V} \|_F$,
   
   $V_{j+1} = \tilde{V} / h_{j+1,j}$.
   
EndFor.
Block GMRES Algorithm

The definition of BGMRES is equivalent to

\[ X_k = X_0 + Z_k \]

\[ \| R_0 - A Z_k \|_F = \min_{Z \in \mathbb{K}_k(A,R_0)} \| R_0 - A Z \|_F. \]  

(8)

Thus BGMRES method proceeds as follows

**ALGORITHM (BGMRES)**

1. Choose \( X_0 \in \mathbb{C}^{N \times s} \) and compute \( R_0 = B - AX_0 \).
2. \( R_0 = V_1 H_{1,0} \) (The QR factorisation of \( R_0 \))
3. For \( j=1, \ldots, k \), do
   - construct \( V_j \) and \( \tilde{H}_j \) by block Arnoldi.
4. Solve the least squares problem :
   \[ Y_k = \text{arg} \min_{Y \in \mathbb{C}^{k_s \times s}} \| R_0 - A V_k Y \|_F, \]
5. The approximate solution is \( X_k = X_0 + V_k Y_k \).
GMRES polynomial

If we denote by $K_k$ the Krylov matrix $K_k = [r_0, \ldots, A^{k-1} r_0]$, then the GMRES polynomial is defined by

$$p_{GMRES}^k(\xi) = \frac{1}{\det(K_k^T A^T A K_k)} \begin{vmatrix} 1 & \xi & \cdots & \xi^k \\ (A r_0, r_0) & (A r_0, A r_0) & \cdots & (A r_0, A^k r_0) \\ \vdots & \vdots & \cdots & \vdots \\ (A^k r_0, r_0) & (A^k r_0, A r_0) & \cdots & (A^k r_0, A^k r_0) \end{vmatrix}.$$

GMRES

- We have $p_{GMRES}^k(0) = 1$ and $r_{GMRES}^k = p_{GMRES}^k(A)r_0$.
- $p_{GMRES}^k(\xi) = 1 - (\xi \xi^2 \cdots \xi^k)(K_k^T A^T A K_k)^{-1} K_k^T A^T r_0$. 
Convergence results for GMRES

Diagonalizable matrices

If the matrix $A$ is diagonalizable $A = X \Lambda X^{-1}$, and $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. It is well known that

$$\|r_k\| \leq \|r_0\| \|X\| \|X^{-1}\| \min_{p \in P_k, p(0)=1} \max_{i=1,\ldots,n} |p(\lambda_i)|,$$

where $\lambda_i$ is an eigenvalue of $A$.

If $r_k = p(A) r_0$, then

$$\|r_k\| = \|X p(\Lambda) X^{-1} r_0\| \leq \kappa(X) \|r_0\| \|p(\Lambda)\|,$$

where $\kappa(X) = \|X\| \|X^{-1}\|$ is the condition number of $X$.

However since the matrix $X$ is not uniquely defined. We can write

$$\frac{\|r_k\|}{\|r_0\|} \leq \inf_X \kappa(X) \min_{p \in P_k, p(0)=1} \max_{i=1,\ldots,n} |p(\lambda_i)|,$$
Diagonalizable matrix

Let us study the possible simplest case \((n = 2, \ k = 1)\). Then \(A = X \text{diag}(\lambda_1, \lambda_2) X^{-1}\) and

\[
X^H X = \begin{pmatrix}
1 & c_1 \\
\frac{1}{c_1} & 1
\end{pmatrix}, \quad \text{with} \quad |c_1| < 1.
\]

The exact residual norms is given by

\[
||r_1||^2 = \frac{(1 - |c_1|^2) |\alpha_1|^2 |\alpha_2|^2 |\lambda_2 - \lambda_1|^2}{|\alpha_1|^2 |\lambda_1|^2 + 2 \text{Re}(c_1 \alpha_1 \lambda_1 \alpha_2 \lambda_2) + |\alpha_2|^2 |\lambda_2|^2},
\]

with \(r_0 = X(\alpha_1, \alpha_2)^T\)
Convergence results for GMRES: Example

In order to obtain the optimal bound for \( \frac{||r_1||}{||r_0||} \) we have to solve an optimization problem.

Diagonalizable matrices: Example

If we assume that all the parameters are reals and \( \lambda_1 \lambda_2 > 0 \), we obtain

\[
\frac{||r_1||}{||r_0||} \leq \sqrt{1 - |c_1|^2} \frac{|\lambda_2 - \lambda_1|}{|\lambda_1 - 2c_1\sqrt{\lambda_1\lambda_2} + \lambda_2|}.
\]

It is obvious that this bound refines the classical ones.

\[
\frac{||r_1||}{||r_0||} \leq \frac{\sqrt{1 + |c_1|}}{\sqrt{1 - |c_1|}} \frac{|\lambda_2 - \lambda_1|}{|\lambda_1| + |\lambda_2|}.
\]

If \( c_1 = 0 \), the

\[
\frac{||r_1||}{||r_0||} \leq \frac{|\lambda_2 - \lambda_1|}{|\lambda_1| + |\lambda_2|}.
\]
Normal matrices \((n \geq 2, \ k = 1)\)

Then \(A = X \ diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \ X^{-1}\) and \(X^H \ X = I\)

\[
||r_1||^2 = \frac{\sum_{1 \leq i < j \leq n} |\alpha_i|^2 |\alpha_j|^2 |\lambda_i - \lambda_j|^2}{|\alpha_1|^2 |\lambda_1|^2 + \ldots + |\alpha_n|^2 |\lambda_n|^2},
\]

with \(r_0 = X(\alpha_1, \ldots, \alpha_n)^T\).

\[
\frac{||r_1||^2}{||r_0||^2} = \frac{\sum_{1 \leq i < j \leq n} \beta_i \beta_j |\lambda_i - \lambda_j|^2}{\sum_{i=1}^n \beta_i |\lambda_i|^2} = F_1(\beta_1, \ldots, \beta_n),
\]

where \(\beta_i = \frac{|\alpha_i|^2}{\sum_{j=1}^n |\alpha_j|^2}\).

We have \(0 \leq \beta_i \leq 1\) and \(\sum_{j=1}^n \beta_j = 1\).
\[(n \geq 2, \ k = 1)\]

\[
\frac{\|r_1\|^2}{\|r_0\|^2} \leq F_1(\beta^*),
\]

\[
F_1(\beta^*) = \max_{\sum_{i=1}^{n} \beta_i = 1} f_1(\beta)
\]

If all eigenvalues are reals, we have

\[
F_1(\beta^*) = \left( \frac{|\lambda_{i_2} - \lambda_{i_1}|}{|\lambda_{i_1}| + |\lambda_{i_2}|} \right)^2 = \delta,
\]

and \(\beta_j^* = 0\) if \(j \notin \{i_1, i_2\}\),

\[
\beta_{i_1}^* = \frac{1}{2}(1 - \sqrt{\delta}) \quad \text{and} \quad \beta_{i_2}^* = \frac{1}{2}(1 + \sqrt{\delta}).
\]

Consequently \(r_2 = 0\).
If one of the eigenvalues is complex, we have $C \leq \frac{4}{\pi}$.

$$F_1(\beta^*) = \left( \frac{|\lambda_{i2} - \lambda_{i1}|}{|e^{i\theta_{i1}} \lambda_{i2} - e^{i\theta_{i2}} \lambda_{i1}|} \right)^2 \leq C^2 \left( \frac{|\lambda_{i2} - \lambda_{i1}|}{|\lambda_{i1}| + |\lambda_{i2}|} \right)^2.$$ 

Exemple : Let us consider the following matrix, $\Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 + i & 0 \\ 0 & 0 & 3 \end{pmatrix}$,

$\beta_1^* = \frac{3}{13}, \beta_2^* = \frac{6}{13}$ and $\beta_3^* = \frac{4}{13}$.

The optimal choice is given by $e^{i\theta_{i1}} = \frac{3 + 2i}{\sqrt{13}}, e^{i\theta_{i2}} = \frac{2 - 3i}{\sqrt{13}}$, and $e^{i\theta_{i3}} = \frac{-2 + 3i}{\sqrt{13}}$. We have also

$$\sqrt{\delta^*} = \begin{vmatrix} e^{i\theta_{i1}} & \lambda_1 \\ e^{i\theta_{i2}} & \lambda_2 \\ 1 & \lambda_1 \\ 1 & \lambda_2 \end{vmatrix} = \begin{vmatrix} e^{i\theta_{i1}} & \lambda_1 \\ e^{i\theta_{i3}} & \lambda_3 \\ 1 & \lambda_1 \\ 1 & \lambda_3 \end{vmatrix} = \begin{vmatrix} e^{i\theta_{i2}} & \lambda_2 \\ e^{i\theta_{i3}} & \lambda_3 \\ 1 & \lambda_2 \\ 1 & \lambda_3 \end{vmatrix}.$$

We have $\frac{\|r_1\|}{\|r_0\|} \leq \frac{1}{\sqrt{13}}$ and $r_2 \neq 0$.
(\(n \geq 2, k = 2\))

\(k = 2\), we obtain

\[
\frac{||r_2||^2}{||r_0||^2} = \frac{\sum_{i,j,k} \beta_i \beta_j \beta_k |\lambda_j - \lambda_i|^2 |\lambda_k - \lambda_j|^2 |\lambda_k - \lambda_i|^2}{\sum_{i,j} \beta_i \beta_j |\lambda_i|^2 |\lambda_j|^2 |\lambda_j - \lambda_i|^2}.
\]

We assume that \(\{1, 2, 3, 4\} \subset Sp(A) \subset [1, 2] \cup [3, 4]\)

1. We have \(\beta_1^* = \frac{3}{5} - \beta_4^*, \beta_2^* = \frac{3}{5} - 3\beta_4^*, \beta_3^* = -\frac{1}{5} + 3 \beta_4^*, \) and \(\beta_4^* \in \left[\frac{1}{15}, \frac{1}{5}\right]\).

2. \[
\frac{||r_1||}{||r_0||} \leq \frac{3}{5}
\]

3. \[
\frac{||r_2||}{||r_0||} \leq \frac{1}{5}
\]

\(r_3 \neq 0\), but \(r_4 = 0\).
Influence of the initial residual

**Theorem**

Let us assume that the columns of $X$ are normalized i.e. $\|X_i\| = 1$ where $X = [X_1, \ldots, X_n]$. If we expand $r_0$ in the eigen-basis $r_0 = X \alpha$, then

$$\|r_k\| \leq \left( \sum_{i=1}^{n} |\alpha_i| \right) \min_{p \in \tilde{P}_k} \max_{\lambda \in \sigma(A)} |p(\lambda)|.$$

If the matrix $A$ is normal ($X^H X = I$), then we have

$$\|r_k\| \leq \left( \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} \right) \min_{p \in \tilde{P}_k} \max_{\lambda \in \sigma(A)} |p(\lambda)|.$$

where $\tilde{P}_k$ is the set of polynomials of degree less or equal to $k$, such that $p(0) = 1$. 

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Proof

Let $K_k$ be the Krylov matrix whose columns are $r_0, Ar_0, \ldots, A^{k-1} r_0$, we have:

[HS, Habilitation Thesis]

If $\|r_k\| \neq 0$ then $\|r_k\|^2 = \frac{\det(K_{k+1}^H K_{k+1})}{\det(K_k^H A^H A K_k)} = \frac{1}{e_1^T (K_{k+1}^H K_{k+1})^{-1} e_1}$.

- [1] I. Ipsen, Expressions and bounds for the Gmres Residual, BIT, 38 (1998) 101-104 $\|r_k\| = \frac{1}{\|e_1^T (K_{k+1}^\dagger)\|}$.

- It is not obvious how the expressions for normal matrices in [1] compares to existing polynomial bounds (Min-Max).
Ipsen Decomposition

\[ \|r_k\| = \frac{1}{(X D_\alpha V_{k+1}^*)^H e_1}, \]

\[ \|r_k\|^2 = \frac{1}{e_1^T \left( V_{k+1}^H D_\alpha^H X^H X D_\alpha V_{k+1} \right)^{-1} e_1}, \]

where

\[ D_\alpha = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n), \quad V_k = \begin{pmatrix}
1 & \lambda_1 & \cdots & \lambda_1^{k-1} \\
1 & \lambda_2 & \cdots & \lambda_2^{k-1} \\
\vdots & \vdots & & \vdots \\
1 & \lambda_n & \cdots & \lambda_n^{k-1}
\end{pmatrix}. \]
Comparison

Theorem

Introduce the function

\[ F_k(t) = \frac{1}{e_1^H (V_{k+1}^H D_t V_{j+1})^{-1} e_1}, \]  

where \( t = (t_1, \ldots, t_n)^T \).

If \( \rho = (\rho_1, \ldots, \rho_n)^T \) where \( \rho_i = \frac{|\alpha_i|}{\sum_{j=1}^{n} |\alpha_j|} \), then

\[ \|r_k\|^2 \leq \left( \sum_{i=1}^{n} |\alpha_i|^2 \right) F_k(\rho). \]

Let the matrix \( A \) in addition be normal, then

\[ \|r_k\|^2 = \left( \sum_{i=1}^{n} |\alpha_i|^2 \right) F_k(\beta), \]

where \( \beta = (\beta_1, \ldots, \beta_n)^T \) and \( \beta_i = \frac{|\alpha_i|^2}{\sum_{j=1}^{n} |\alpha_j|^2} \).
the saddle point problem with multiple right-hand sides

Many problems in Science and Engineering require the solution of the saddle point problem with multiple right-hand sides.

\[
\begin{pmatrix}
A & B^T \\
\epsilon B & O
\end{pmatrix}
\begin{pmatrix}
X \\
Y
\end{pmatrix} =
\begin{pmatrix}
F \\
\epsilon G
\end{pmatrix},
\]

(12)

Where \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite matrix and \( B^T \in \mathbb{R}^{n \times m} \) has full column rank, with \( X \in \mathbb{R}^{n \times s} \), \( Y \in \mathbb{R}^{m \times s} \) and \( F \in \mathbb{R}^{n \times s} \), \( G \in \mathbb{R}^{m \times s} \).
Convergence analysis of the global GMRES method

In this subsection, we recall some convergence results for the global GMRES method. Let \( A = ZDZ^{-1} \), where \( D \) is the diagonal matrix whose elements are the eigenvalues \( \lambda_1, \ldots, \lambda_{n+m} \), and \( Z \) is the eigenvector matrix.

Let the initial residual \( R_0 \) be decomposed as \( R_0 = Z \beta \) where \( \beta \) is an \((n + m) \times s\) matrix whose columns are denoted by \( \beta^{(1)}, \ldots, \beta^{(s)} \). Let \( R_k = B - AX_k \) be the \( k \)th residual obtained by the global GMRES when applied to (12). Then we have

\[
\|R_k\|_F^2 \leq \frac{\|Z\|_2^2}{e_1^T (V_{k+1}^T \tilde{D} V_{k+1})^{-1} e_1},
\]

where

\[
\tilde{D} = \begin{pmatrix}
\sum_{i=1}^{s} |\beta_1^{(i)}|^2 \\
\vdots \\
\sum_{i=1}^{s} |\beta_{n+m}^{(i)}|^2
\end{pmatrix}
\]

and \( V_{k+1} = \begin{pmatrix}
1 & \lambda_1 & \ldots & \lambda_k^k \\
\vdots & \vdots & \ddots & \vdots \\
1 & \lambda_{n+m} & \ldots & \lambda_{n+m}^k
\end{pmatrix} \)

The coefficients \( \beta_1^{(i)}, \ldots, \beta_{n+m}^{(i)} \) are the components of the vector \( \beta^{(i)} \) and \( e_1 \) is the first unit vector of \( \mathbb{R}^{k+1} \).
Preconditioning

In this following section we present the preconditioner $\mathcal{P}_p$, for solving saddle point problems with multiple right-hand sides (12). Now we propose the preconditioner $\mathcal{P}_p$ for solving saddle point problems with multiple right-hand sides (12)

$$
\mathcal{P}_p = \left( \begin{array}{cc}
A & B^T \\
\epsilon B & \alpha Q 
\end{array} \right), \text{ with } \mathcal{P}_p^{-1} A \chi = \mathcal{P}_p^{-1} B.
$$

(15)

Where $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m \times n}$ has a full row rank and $Q$ is an approximation of Shur complement $S = -BA^{-1}B^T$ and $\alpha > 0$. 

Preconditioner factorization

The preconditioner has the block-triangular factorization

\[
\mathcal{P}_p = \begin{pmatrix}
A & B^T \\
\epsilon B & \alpha Q
\end{pmatrix} = \begin{pmatrix}
I & O \\
\epsilon BA^{-1} & I
\end{pmatrix} \begin{pmatrix}
A & O \\
O & \tilde{S}
\end{pmatrix} \begin{pmatrix}
I & A^{-1}B^T \\
O & I
\end{pmatrix},
\] (16)

where \( \tilde{S} = (\alpha Q - \epsilon BA^{-1}B^T) \).

If \( \epsilon = 1 \), the block \( \tilde{S} \) is positive definite matrix for all \( \alpha \lambda_{\min}(Q) > \lambda_{\max}(BA^{-1}B^T) \).

Thus the inverse of the preconditioned matrix \( \mathcal{P}_p \) is given by the following equality

\[
\mathcal{P}_p^{-1} = \begin{pmatrix}
A & B^T \\
\epsilon B & \alpha Q
\end{pmatrix}^{-1} = \begin{pmatrix}
I & -A^{-1}B^T \\
O & I
\end{pmatrix} \begin{pmatrix}
A^{-1} & O \\
O & \tilde{S}^{-1}
\end{pmatrix} \begin{pmatrix}
I & O \\
BA^{-1} & I
\end{pmatrix}. \] (17)
Preconditioned matrix

If $\epsilon = -1$, the preconditioned matrix $P_p^{-1}A$ can be rewritten as follow

$$P_p^{-1}A = \left( \begin{array}{cc} A & B^T \\ -B & \alpha Q \end{array} \right)^{-1} \left( \begin{array}{cc} A & B^T \\ -B & O \end{array} \right) = \left( \begin{array}{cc} I & K_1 \\ O & K_2 \end{array} \right),$$

(18)

where $\tilde{S} = (\alpha Q + BA^{-1}B^T)$, $K_1 = A^{-1}B^T - A^{-1}B^T \tilde{S}^{-1}BA^{-1}B^T$ and $K_2 = \tilde{S}^{-1}BA^{-1}B^T$. 

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Algorithm 2: The precondioned Global GMRES

1: $\mathcal{P}_p V_1 = R_0$, $V_1 = V_1 / \|V_1\|_F$
2: for $j = 1, 2, ..., k$ do;
3: $\mathcal{P}_p W := AV_j$;
4: for $i = 1, 2, ..., j$ do;
5: $H_{i,j} = \langle W, V_i \rangle_F$;
6: $W = W - H_{i,j} V_i$;
7: end;
8: $H_{j+1,j} = \|W\|_F$;
9: $V_{j+1} = W / H_{j+1,j}$;
10: Solve the linear system $H_{k,k} y = \beta e_1$ for $y$;
11: Set $\mathcal{X}_k = \mathcal{X}_0 + V_k \diamond y$ and $R_k = B - A \mathcal{X}_k$;
12: end for
The precondioned Global GMRES

At each step of applying the preconditioner $P_p$ inside the GMRES algorithm, we need to solve the system 1 and 3 of algorithm 2. For a given matrix $V = [V_1; V_2]$ where $V_1 \in \mathbb{R}^{n \times s}$ and $V_2 \in \mathbb{R}^{m \times s}$. Let $Z = [Z_1; Z_2]$, where $Z_1 \in \mathbb{R}^{n \times s}$ and $Z_2 \in \mathbb{R}^{m \times s}$.

\[
\begin{pmatrix}
A & B^T \\
-B & \alpha Q
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
= 
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix}.
\tag{19}
\]

We can solve (19) by using the following algorithm.

---

**Algorithm 3** :

1 : Solve $\underbrace{\left(A + \frac{1}{\alpha}B^TQ^{-1}B\right)}_{A_{\alpha}} Z_1 = V_1 - \frac{1}{\alpha}B^TQ^{-1}V_2$ ;

2 : Compute $Z_2 = \frac{1}{\alpha}Q^{-1}(V_2 + BZ_1)$ ;

The matrix $A_{\alpha}$ is symmetric positive definite. Therefore, we can solve the system with the coefficient matrix $A_{\alpha}$ by the preconditioned global CG method or by the preconditioned global MINRES method inexactly.
### Numerical results

**Table 3**: Numerical results for multiple right-hand sides with global approach.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$P_p$</th>
<th>$P_{VPSS}$</th>
<th>$P_T$</th>
<th>$P_D$</th>
</tr>
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<tbody>
<tr>
<td>$10^{-5}$</td>
<td>IT 4</td>
<td>IT 72</td>
<td>IT 83</td>
<td>IT 1</td>
</tr>
<tr>
<td></td>
<td>CPU 1.46</td>
<td>CPU 4.85</td>
<td>CPU 2.81</td>
<td>CPU 3</td>
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<td></td>
<td>RES 1.25e-04</td>
<td>RES 1.08e-05</td>
<td>RES 9.10e-07</td>
<td>RES 1.09e-06</td>
</tr>
<tr>
<td></td>
<td>ERR 6.06e-03</td>
<td>ERR 6.27e-05</td>
<td>ERR 7.68e-06</td>
<td>ERR 9.23e-06</td>
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<tr>
<td>$10^{-4}$</td>
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<td>IT 75</td>
<td>IT 94</td>
<td>IT 1</td>
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<td>CPU 1.83</td>
<td>CPU 5.28</td>
<td>CPU 3.24</td>
<td>CPU 3</td>
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<tr>
<td></td>
<td>RES 3.93e-06</td>
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<td>ERR 9.23e-06</td>
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<tr>
<td>$10^{-3}$</td>
<td>IT 14</td>
<td>IT 74</td>
<td>IT 87</td>
<td>IT 1</td>
</tr>
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<td>CPU 1.58</td>
<td>CPU 5.03</td>
<td>CPU 3.64</td>
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<tr>
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<td>ERR 7.72e-06</td>
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<tr>
<td>$10^{-2}$</td>
<td>IT 27</td>
<td>IT 74</td>
<td>IT 78</td>
<td>IT 1</td>
</tr>
<tr>
<td></td>
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<td>CPU 5.09</td>
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<tr>
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<td>CPU 4.35</td>
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Thank you for your attention