The third conference on Numerical Analysis and Scientific Computation with Applications

The inverse and variational data assimilation problem on finding the heat flux in the sea thermodynamics model

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Contents

Features of the model

- Forward model
  (*V.Zalesny, N.Diansky, A.Gusev*)
  - Free surface sigma-coordinate ocean model;
  - Multi-component splitting, modular model structure.
- Adjoint block
  (*V.Agoshkov, V. Shutyaev, E.Parmuzin*)
  - Exact adjoint for temperature splitting subsystem/module.
- Numerical algorithms
  - Implicit time stepping;
  - Iterative cost function minimization.

Mathematical model

$$
\begin{cases}
\dfrac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g\mathbf{grad}\xi + A_u\vec{u} + (A_k)^2\vec{u} = \vec{f} - \dfrac{1}{\rho_0}\mathbf{grad}P_a - \\
-\dfrac{g}{\rho_0}\mathbf{grad}\displaystyle\int_0^z \rho_1(T,S)dz', \\
\dfrac{\partial\xi}{\partial t} - m\dfrac{\partial}{\partial x}(\displaystyle\int_0^H \Theta(z)udz) - m\dfrac{\partial}{\partial y}(\displaystyle\int_0^H \Theta(z)\dfrac{n}{m}vdz) = f_3, \\
\dfrac{dT}{dt} + A_T T = f_T, \ \dfrac{dS}{dt} + A_S S = f_S,
\end{cases}
$$

where $\rho_1(T,S) = \rho_0\beta_T(T - T^{(0)}) + \rho_0\beta_S(S - S^{(0)}) + \gamma\rho_0\beta_{TS}(T,S) + f_P$, $\vec{f} = (f_1, f_2)$, f_T, f_S, and f_P are given functions of internal sources, $g = const > 0$, ρ_0, $T^{(0)}$, and $S^{(0)}$ are the unperturbed water density, temperature, and salinity, β_T, β_S are coeficients (assumed to be constant), $\beta_{TS}(T,S)$, P_a, $f_3 \equiv f_3(x,y,\zeta,t) \equiv f_3(x,y,t)$ are given functions, and γ is a numerical parameter.

Boundary conditions on the surface

$$\left(\int_0^H \Theta \vec{u} dz\right)\vec{n} + \beta_0 m_{op}\sqrt{gH}\,\xi = m_{op}\sqrt{gH}\,d_s \text{ on } \partial\Omega,$$

$$U_n^{(-)}u - \nu\frac{\partial u}{\partial z} - k_{33}\frac{\partial}{\partial z}A_k u = \tau_x^{(a)}\Big/\rho_0,$$

$$A_k u = 0, \quad A_k v = 0,$$

$$U_n^{(-)}T - \nu_T\frac{\partial T}{\partial z} + \gamma_T(T - T_a) = Q_T + U_n^{(-)}d_T,$$

$$U_n^{(-)}S - \nu_S\frac{\partial S}{\partial z} + \gamma_S(S - S_a) = Q_S + U_n^{(-)}d_S.$$

where $\tau_x^{(a)}$, $\tau_y^{(a)}$ are the wind stress components along the Ox and Oy axes at $z = 0$, γ_T, γ_S, T_a, S_a, Q_T, Q_S, d_T, and d_S are given functions, $U_n|_{z=0} = -w|_{z=0}$, and $w = w(u, v)$ is defined by the formula which is derived by integrating the continuity equation with respect to $z' \in (z, H)$.

Splitting method. Temperature

Step 1. We consider the system:

$$\begin{cases} T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\ T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\ T_j \equiv T \text{ on } D \times (t_{j-1}, t_j). \end{cases}$$

Splitting method. Salinity

Step 2.

$$
\begin{cases}
S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j), \\
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

Splitting method. Circulation and sea level

$$
\begin{cases}
\underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad} G - \dfrac{1}{\rho_0} \mathbf{grad} \left(P_a + g \displaystyle\int_0^z \rho_1(\bar{T}, \bar{S}) dz' \right) \\
\xi_t - \mathbf{div} \left(\displaystyle\int_0^H \Theta \underline{u}^{(1)} dz \right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\
\underline{u}^{(1)} = \underline{u}_{j-1}, \ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\
\left(\displaystyle\int_0^H \Theta \underline{u}^{(1)} dz \right) \cdot n + \beta_0 m_{op} \sqrt{gH} \xi = m_{op} \sqrt{gH} d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\
\underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D
\end{cases}
$$

$$
\begin{cases}
\underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\
\underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ при } t = t_{j-1} \text{ in } D, \\
\underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D,
\end{cases}
$$

4D-Var data assimilation

The function obtained by observations processing is the function T_{obs} at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. We consider this function as an approximation to SST data on Ω, i.e. to $T\Big|_{z=0}$. We assume that the function T_{obs} is known only on the part of $\Omega \times (0, \bar{t})$ and we define a support of this function as m_0. The function of full flux Q is an additional unknown function ("control") and we introduce the cost-function in the form:

$$J_\alpha(Q, T) = \frac{1}{2} \int_0^{\bar{t}} \int_\Omega \alpha |Q - Q^{(0)}|^2 d\Omega dt + J_0(T),$$

$$J_0(T) = \frac{1}{2} \int_0^{\bar{t}} \int_\Omega m_0 (T - T_{obs}) R^{-1} (T - T_{obs})^T d\Omega dt.$$

Here $\alpha \equiv \alpha(\lambda, \theta, t)$ is a regularization function (it is possible that $\alpha(\lambda, \theta, t) =$const$\geq 0$) and it may be a dimensional quantity; $Q^{(0)} \equiv Q^{(0)}(\lambda, \theta, t)$ is a given function.

Data assimilation problem. Find the full solution ϕ (i.e. $T, S, \vec{u}, \xi$) of the system and function Q, such that, the functional J_α is minimal on the set of the solutions.

Optimality system

The optimality system obtained consists of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. The system of equations arising from minimization of the functional J_α on the set of the solution of the equations is

$$\begin{cases} (T_1)_t + L_1 T_1 = \mathcal{F}_1, \; T_1 = T_{j-1} \;\text{ at }\; t = t_{j-1} \\ (T_2)_t + L_2 T_2 = \mathcal{F}_2 + BQ_T, \; T_2(t_{j-1}) = T_1(t_j). \end{cases}$$

$$T_2(t_j) \equiv T_j \cong T \quad \text{at} \;\; t = t_j.$$

$$\begin{cases} (T_2^*)_t + L_2^* T_2^* = B^* m_0 R^{-1}(T - T_{obs}), \; T_2^*(t_j) = 0, \\ (T_1^*)_t + L_1^* T_1^* = 0, \; T_1^*(t_j) = T_2^*(t_{j-1}), \end{cases}$$

$$\alpha(Q - Q^{(0)}) + T_2^* = 0.$$

Functions T_2, $Q(t_j)$ are accepted as approximations to functions T, Q of the full solution for the Problem.

Optimality system in Z-coordinate and iterative process

$$\begin{cases} T_t + \frac{1}{2}\left(w_1 \frac{\partial T}{\partial z} + \frac{1}{r^2}\frac{\partial (r^2 w_1 T)}{\partial z}\right) - \frac{1}{r^2}\frac{\partial}{\partial z} r^2 \nu_T \frac{\partial T}{\partial z} = f_T \\ T = T_1(t_j) \\ -\nu_T \frac{\partial T}{\partial z} = Q_T \ \text{ at } \ z = 0, \ \ \nu_T \frac{\partial T}{\partial z} = 0 \ \text{ at } \ z = H \end{cases}$$

$$\begin{cases} -T_t^* - \frac{1}{2}\left(w_1 \frac{\partial T^*}{\partial z} + \frac{\partial (w_1 T^*)}{\partial z}\right) - \frac{1}{r^2}\frac{\partial}{\partial z}\left(r^2 \nu_T \frac{\partial T^*}{\partial z}\right) = m_0 R^{-1}(T - T_{\text{obs}}) \\ T^* = 0 \quad at \ \ t = t_j, \\ -w_1 T^* - \nu_T \frac{\partial T^*}{\partial z} = 0 \ \ at \ \ z = 0, \quad \nu_T \frac{\partial T^*}{\partial z} = 0 \ \ at \ \ z = H, \end{cases}$$

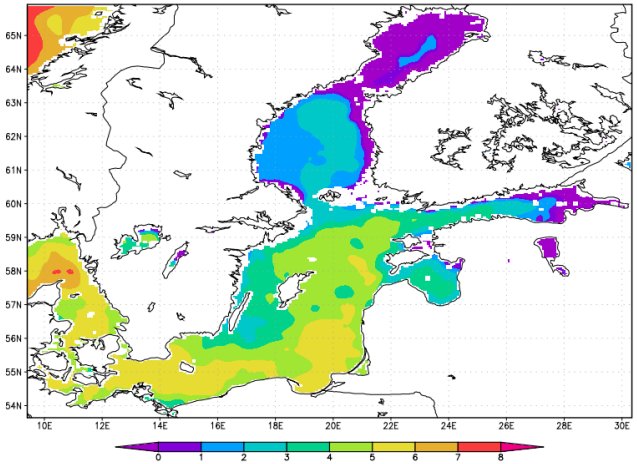
$$Q^{(k+1)} = Q^{(k)} - \gamma_k(\alpha(Q^{(k)} - Q^{(0)}) + T^*) \quad \text{on} \ \ \Omega \times (t_0, t_1).$$

Numerical Experiments

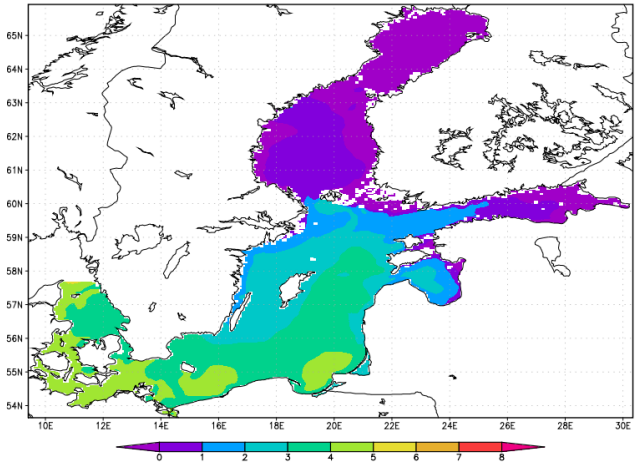
- Numerical experiments have been carried out in the Baltic Sea (model has been developed in the INM RAS).
- Observation data: Daily sea surface temperature (SST) from Danish meteorological Institute.
- The spatial resolution is 0.0625*0.03125 degree.
- The time step is 5 minutes.
- Assimilation intervals: 4 times per day.
- The mean flux $Q^{(0)}$ was taken from the database of NCEP (National Centers for Environmental Prediction).
- Covariance matrix R is calculated based on the statistical properties of observation data.

The observation data assimilation module to assimilate T_{obs} was included into the Baltic Sea thermohydrodynamic model. The experiments start: (A) from 1st of January 2007; (B) from 1st June 2007. Duration of the calculation is 1 months.

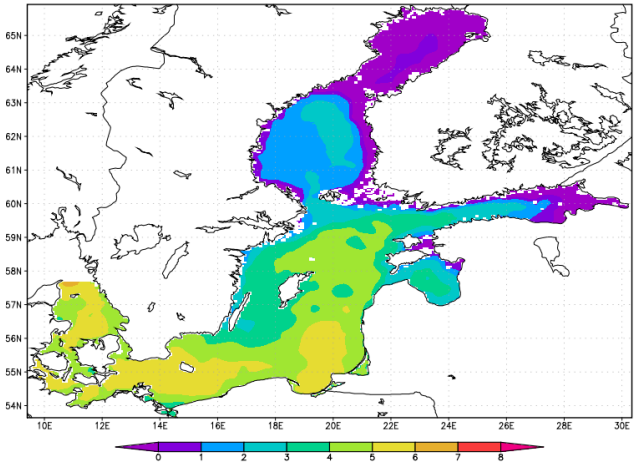
Sea surface temperature (SST). Experiment (A).



(a) **Observation data (average SST)**

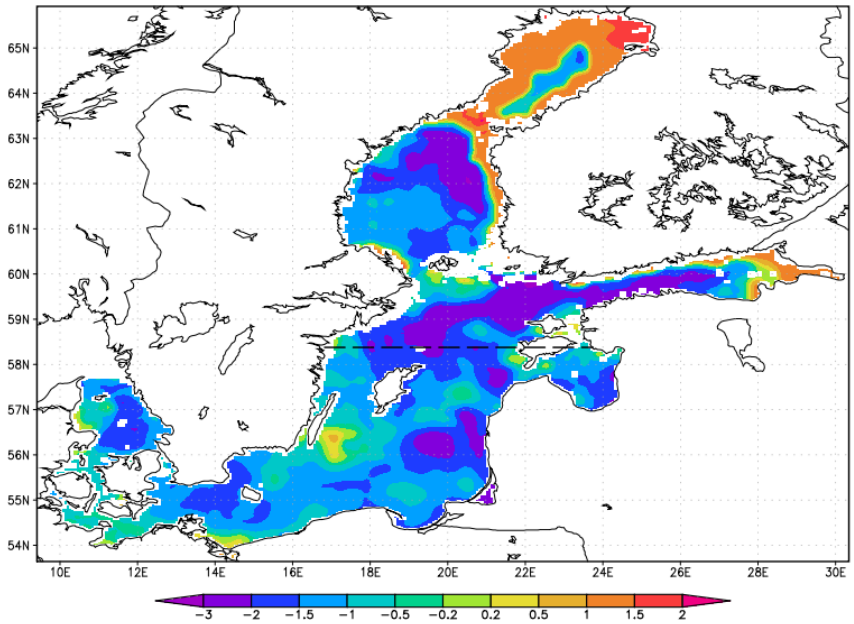


(b) **Calculation without assimilation**

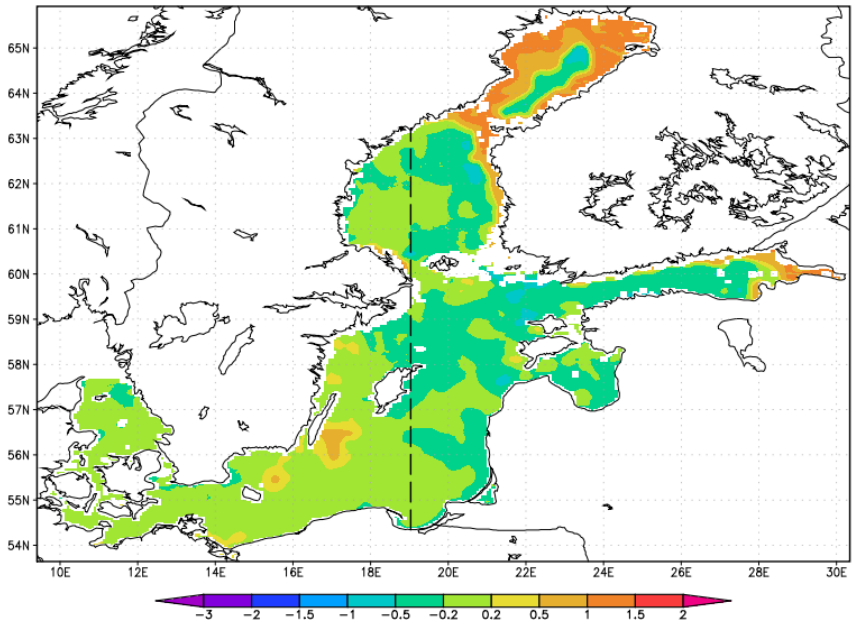


(c) **Calculation with assimilation**

SST. Deviation from observations. Experiment (A).

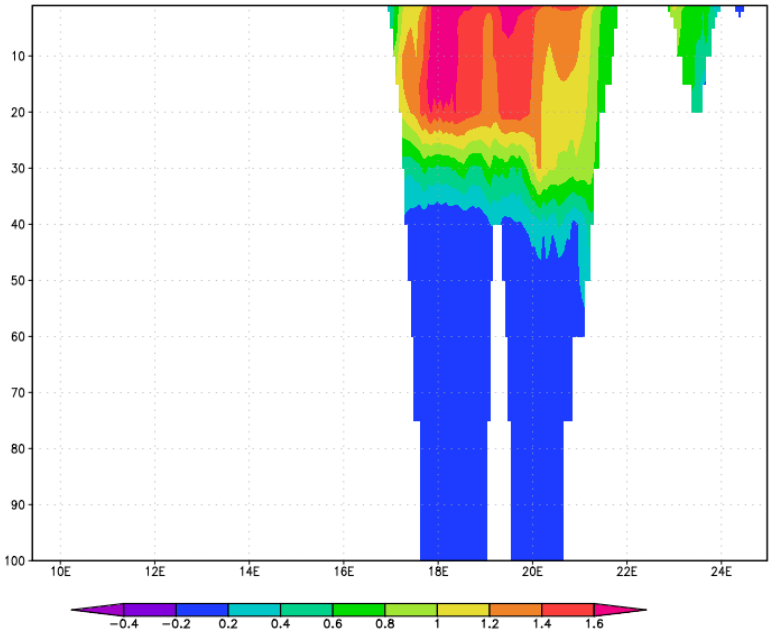


(a) $T_{model} - T_{obs}$

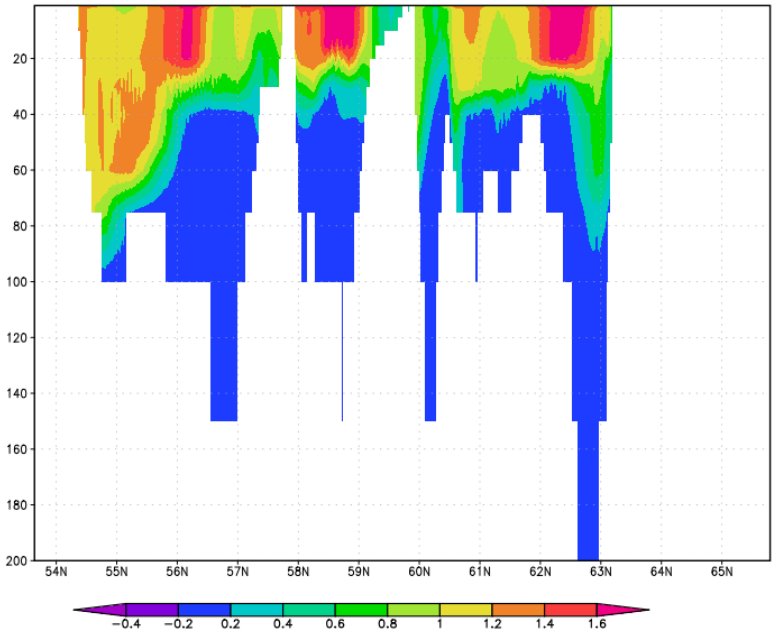


(b) $T_{assim} - T_{obs}$

SST section. Experiment (A).

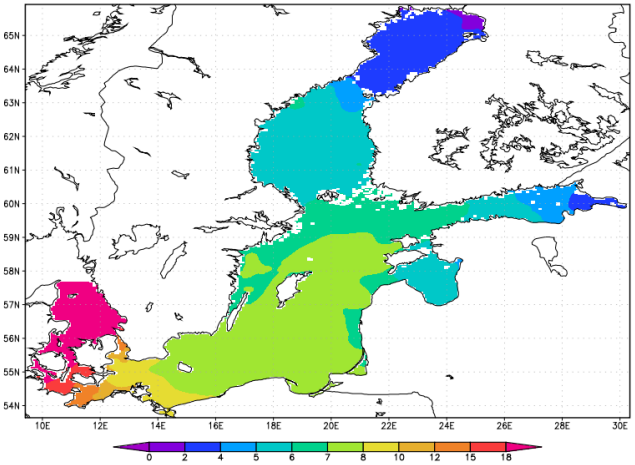


(a) Section on latitude. 58.3° N

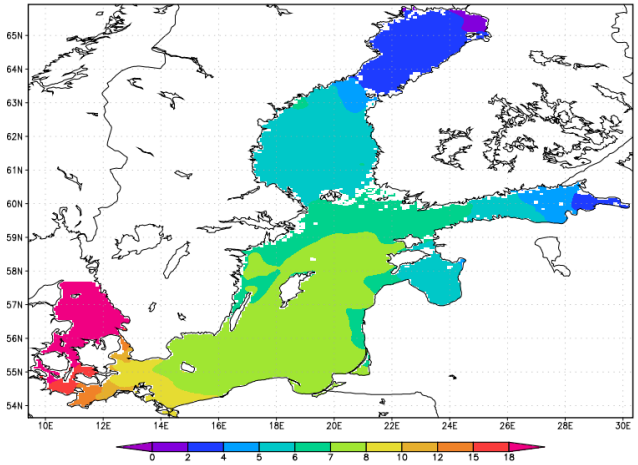


(b) Section on longitude. 19° E

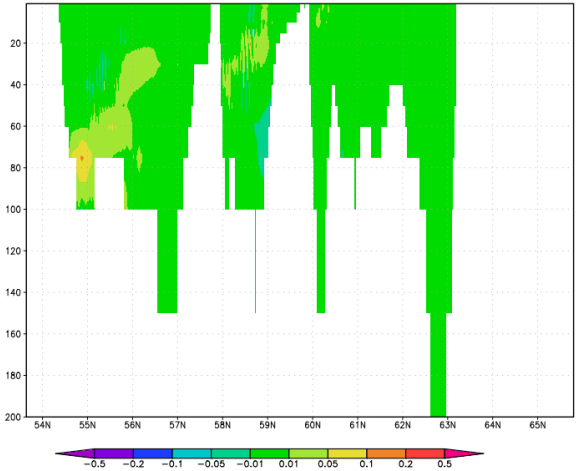
Salinity. Experiment (A).



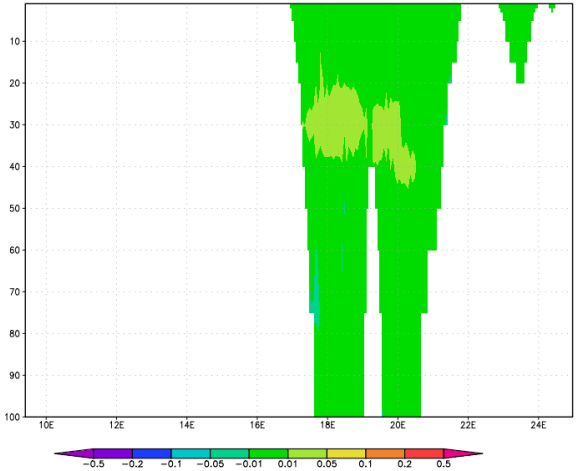
(a) Model salinity



(b) Salinity after SST assimilation

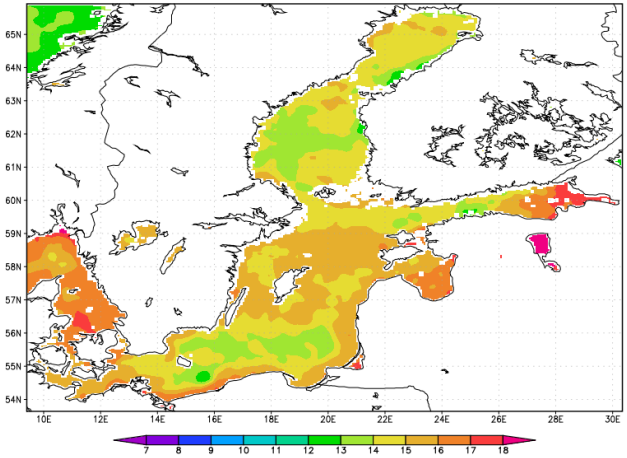


(c) Section on longitude. 19° E

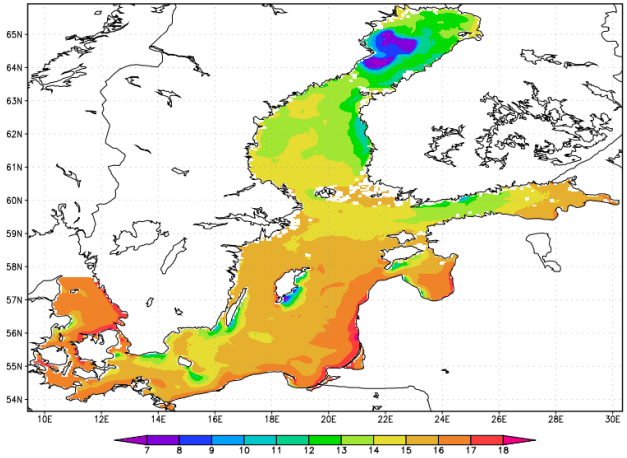


(d) Section on latitude. 58.3° N

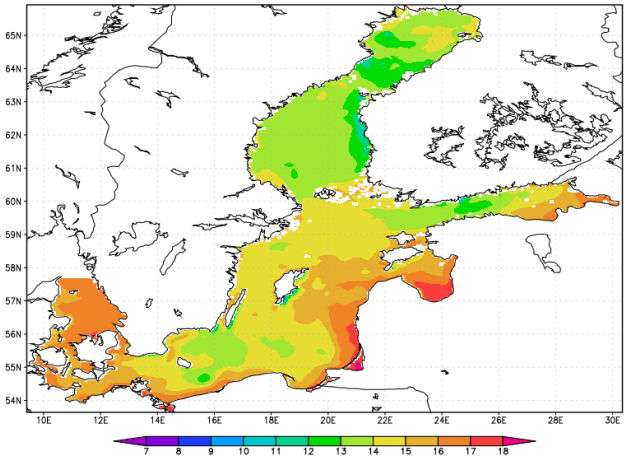
Sea surface temperature. Experiment (B).



(a) **Observation data (average SST)**

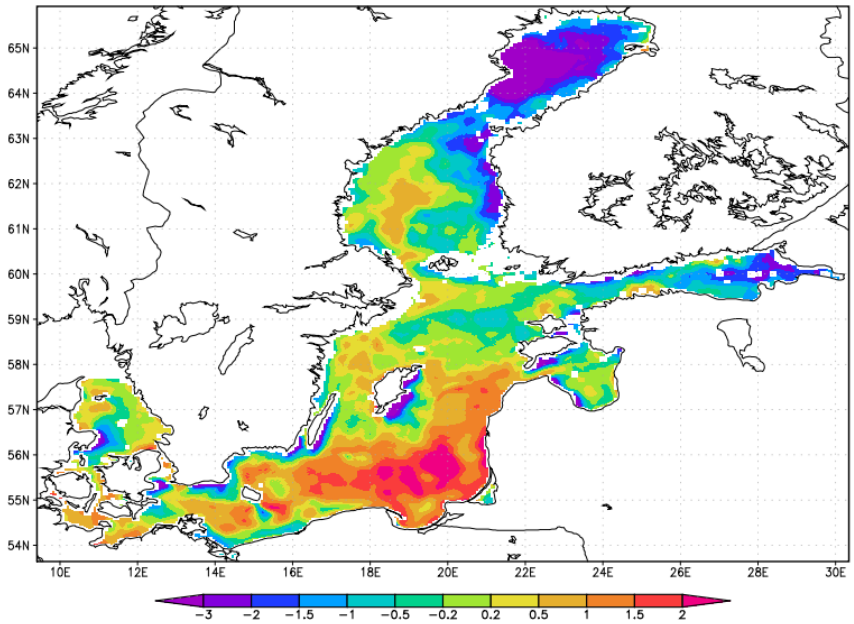


(b) **Calculation without assimilation**

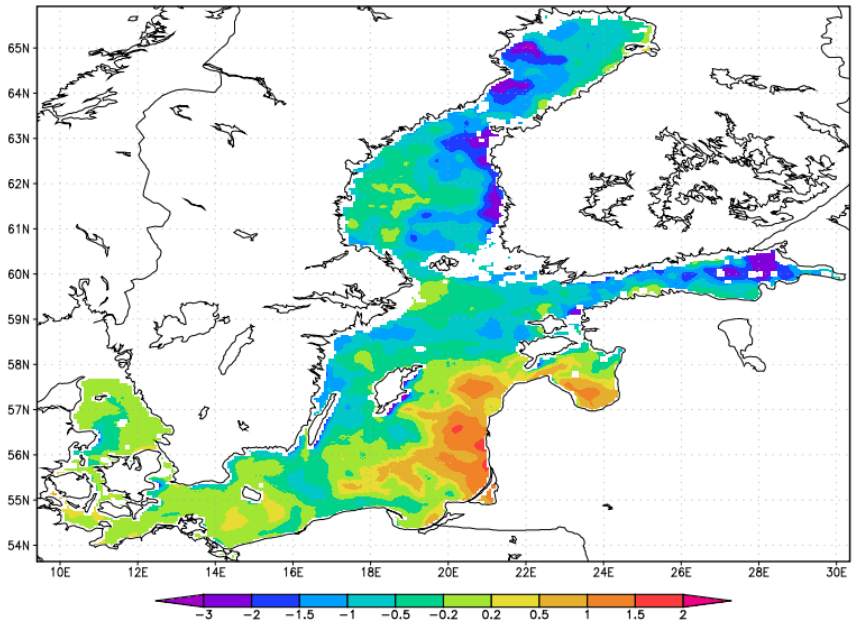


(c) **Calculation with assimilation**

SST. Deviation from observations. Experiment (B).

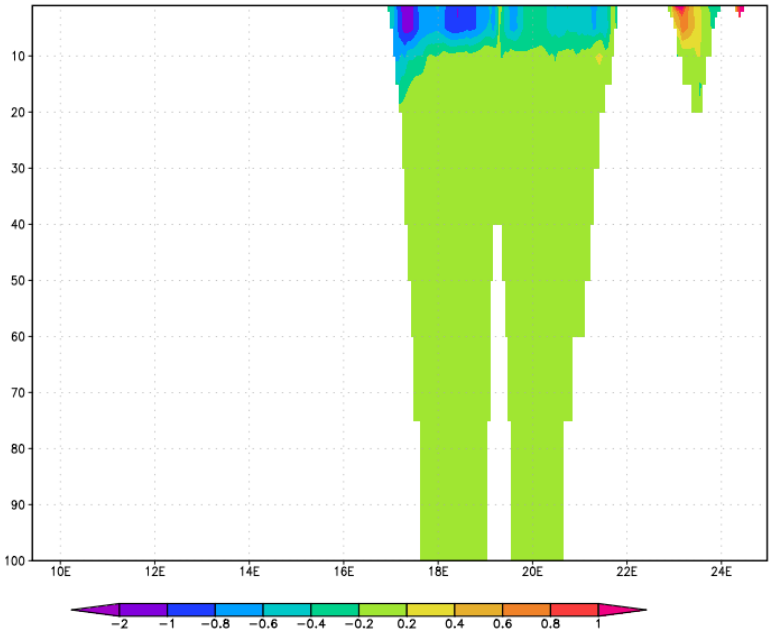


(a) $T_{model} - T_{obs}$

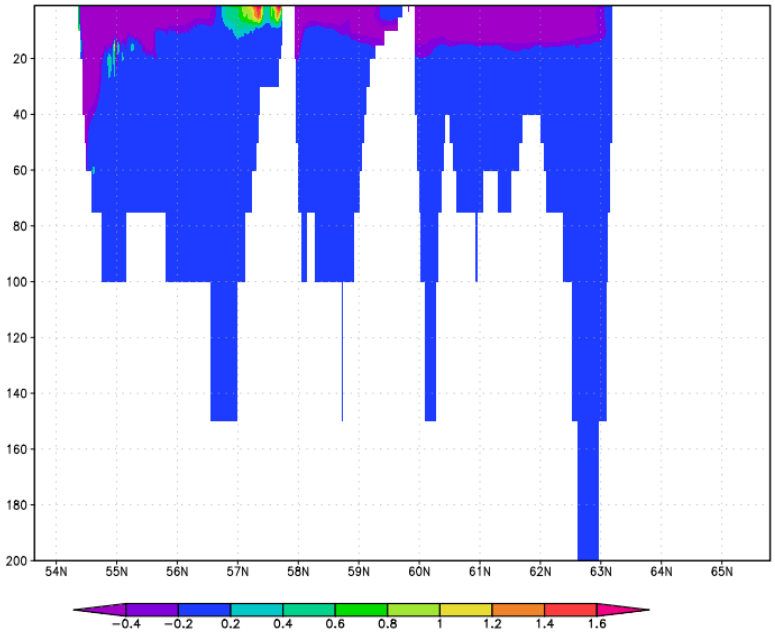


(b) $T_{assim} - T_{obs}$

SST section. Experiment (B).

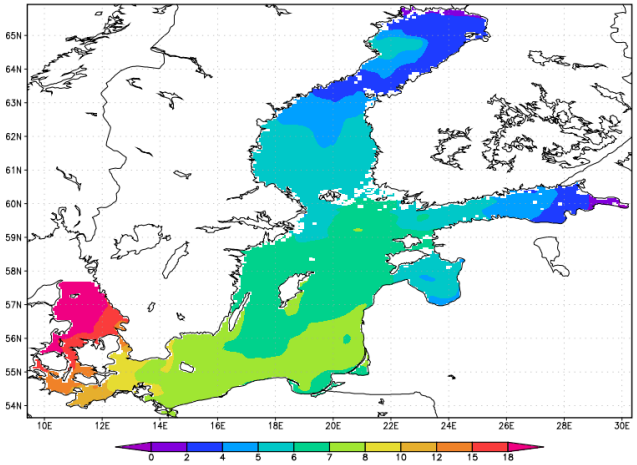


(a) Section on latitude. 58.3° N

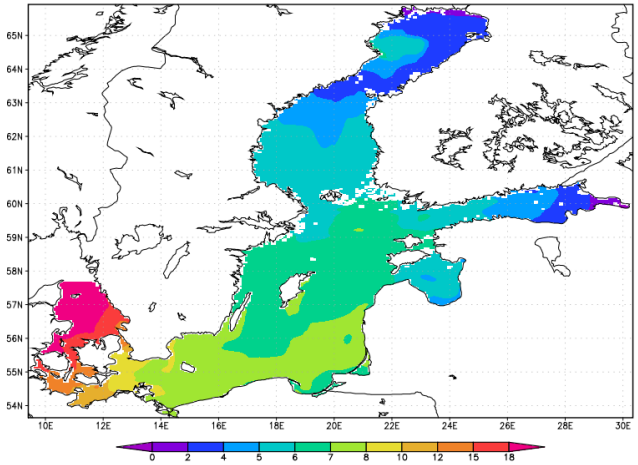


(b) Section on longitude. 19° E

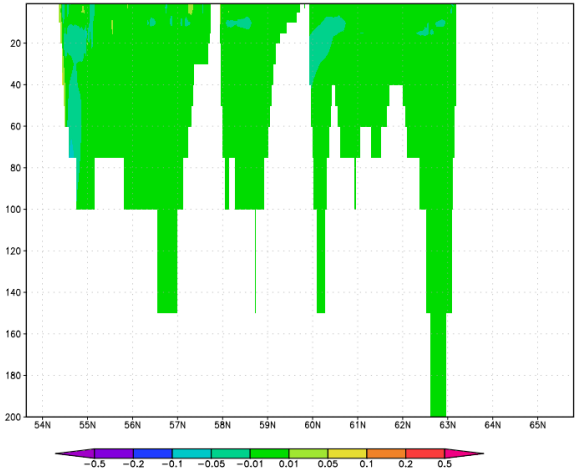
Salinity. Experiment (B).



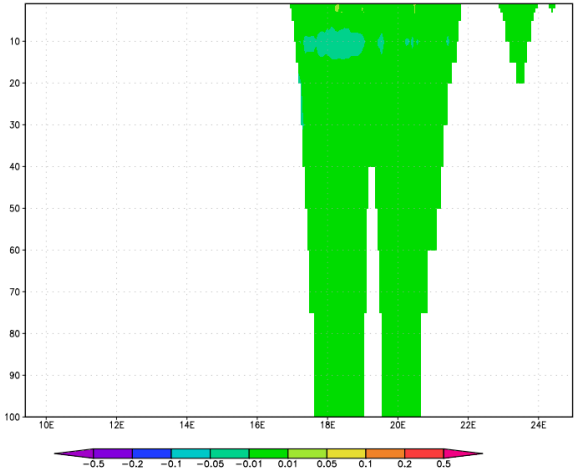
(a) **Model salinity**



(b) **Salinity after SST assimilation**

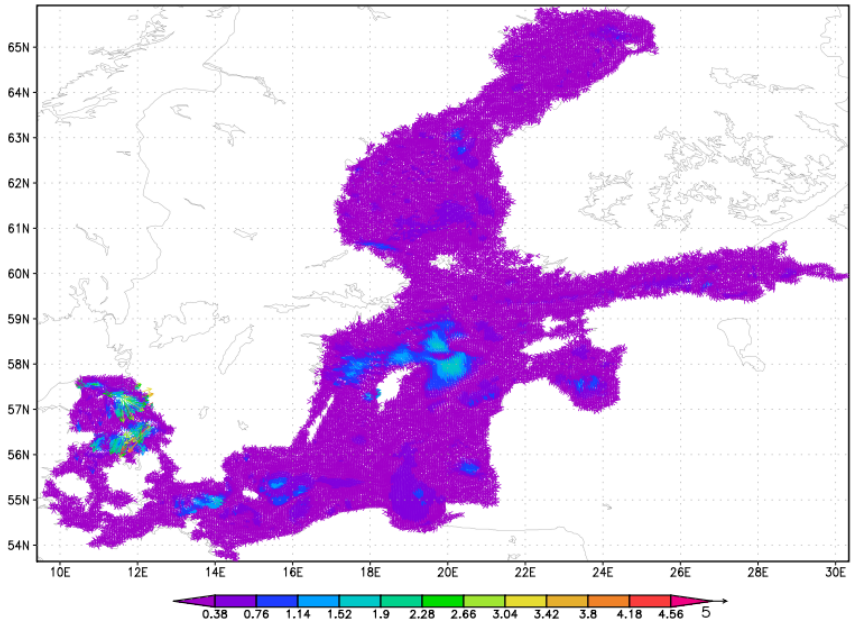


(c) **Section on longitude. 19° E**

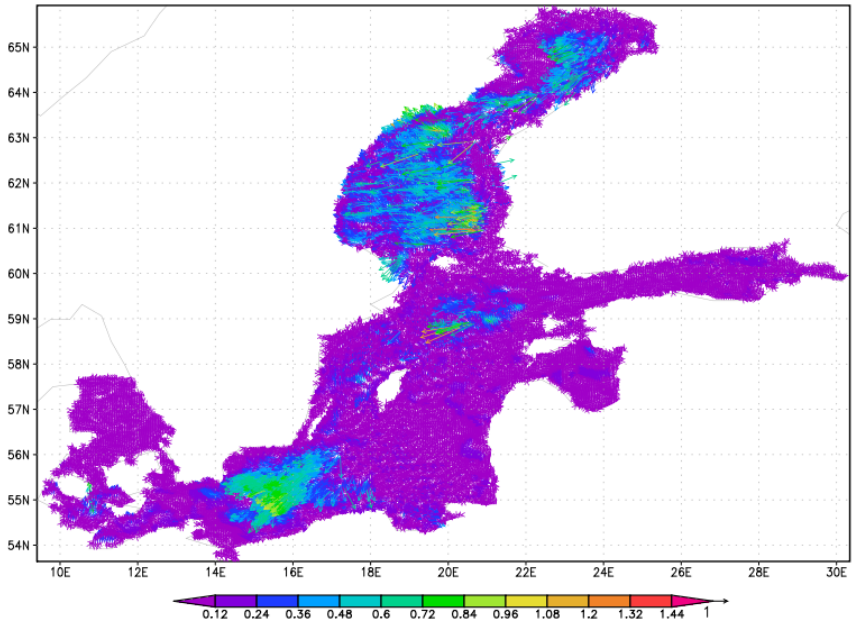


(d) **Section on latitude. 58.3° N**

Difference in velocities



(a) Model vs Assimilation. Experiment (A)



(b) Model vs Assimilation. Experiment (B)

Summary

- The variational data assimilation problem of finding the flux on the sea surface using the observation of SST with covariance matrix in cost function was formulated and studied.
- Algorithms of the numerical solution of data assimilation problem were developed and justified. The assimilation block with covariance matrix was included into 3D hydrodynamics model developed in INM RAS.
- The numerical experiments show that assimilation of SST has a small influence to other components of the full solution, i.e. salinity, velocity etc.

References

1. Zalesny V.B., Gusev A.V., Chernobay S.Yu., Aps R., Tamsalu R., Kujala P., Rytkonen J. The Baltic Sea circulation modelling and assessment of marine pollution // Russ. J. Numer. Analysis and Math. Modelling, 2014, V 29, No. 2, pp. 129-138.
2. Agoshkov V.I., Parmuzin E.I., Zakharova N.B., Zalesny V.B., Shutyaev V.P., Gusev A.V. Variational assimilation of observation data in the mathematical model of the Baltic Sea dynamics // Russ. J. Numer. Anal. Math. Modelling, 2015, V. 30, No. 4, PP. 203-212.
3. Agoshkov V.I., Parmuzin E.I., Zakharova N.B., and Shutyaev V.P. Variational assimilation with covariance matrices of observation data errors for the model of the Baltic Sea dynamics // Russ. J. Numer. Analysis and Math. Modelling, V.33, No.3, pp.149-160, 2018.