The third conference on Numerical Analysis and Scientific Computation with Applications

The inverse and variational data assimilation problem on finding the heat flux in the sea thermodynamics model

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Kalamata, Greece
5 July 2018
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Features of the model

- **Forward model**
  (V. Zalesny, N. Diansky, A. Gusev)
  - Free surface sigma-coordinate ocean model;
  - Multi-component splitting, modular model structure.

- **Adjoint block**
  (V. Agoshkov, V. Shutyaev, E. Parmuzin)
  - Exact adjoint for temperature splitting subsystem/module.

- **Numerical algorithms**
  - Implicit time stepping;
  - Iterative cost function minimization.
Mathematical model

\[
\begin{align*}
\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g\nabla\xi + A_u \vec{u} + (A_k)^2 \vec{u} &= \vec{f} - \frac{1}{\rho_0} \nabla P_a - \\
-\frac{g}{\rho_0} \nabla \int_0^z \rho_1(T, S) dz', \\
\frac{\partial \xi}{\partial t} - m \frac{\partial}{\partial x} \left( \int_0^H \Theta(z) u dz \right) - m \frac{\partial}{\partial y} \left( \int_0^H \Theta(z) \frac{n}{m} v dz \right) &= f_3, \\
\frac{dT}{dt} + A_T T &= f_T, \\
\frac{dS}{dt} + A_S S &= f_S,
\end{align*}
\]

where \( \rho_1(T, S) = \rho_0 \beta_T (T - T^{(0)}) + \rho_0 \beta_S (S - S^{(0)}) + \gamma \rho_0 \beta_{TS} (T, S) + f_P \), \( \vec{f} = (f_1, f_2) \), \( f_T \), \( f_S \), and \( f_P \) are given functions of internal sources, \( g = const > 0 \), \( \rho_0 \), \( T^{(0)} \), and \( S^{(0)} \) are the unperturbed water density, temperature, and salinity, \( \beta_T \), \( \beta_S \) are coefficients (assumed to be constant), \( \beta_{TS}(T, S) \), \( P_a \), \( f_3 \equiv f_3(x, y, \zeta, t) \equiv f_3(x, y, t) \) are given functions, and \( \gamma \) is a numerical parameter.
Boundary conditions on the surface

\[
\begin{bmatrix}
\int_0^H \Theta \tilde{u} \, dz
\end{bmatrix} \vec{n} + \beta_0 m_{op} \sqrt{gH} \xi = m_{op} \sqrt{gH} d_s \text{ on } \partial \Omega,
\]

\[U_n^{(-)} u - \nu \frac{\partial u}{\partial z} - k_{33} \frac{\partial}{\partial z} A_k u = \tau_x^{(a)} / \rho_0,\]

\[A_k u = 0, \quad A_k \nu = 0,\]

\[U_n^{(-)} T - \nu_T \frac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T,\]

\[U_n^{(-)} S - \nu_S \frac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S.\]

where $\tau_x^{(a)}$, $\tau_y^{(a)}$ are the wind stress components along the $Ox$ and $Oy$ axes at $z = 0$, $\gamma_T$, $\gamma_S$, $T_a$, $S_a$, $Q_T$, $Q_S$, $d_T$, and $d_S$ are given functions, $U_n|_{z=0} = -w|_{z=0}$, and $w = w(u, \nu)$ is defined by the formula which is derived by integrating the continuity equation with respect to $z' \in (z, H)$. 

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Nasca 2018

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Splitting method. Temperature

Step 1. We consider the system:

\[ T_t + (\vec{U}, \text{Grad}) T - \text{Div}(\hat{a}_T \cdot \text{Grad} T) = f_T \text{ in } D \times (t_{j-1}, t_j), \]

\[ T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \]

\[ \vec{U}^{(-)}_n T - \nu_T \frac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \vec{U}^{(-)}_n d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \]

\[ \frac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \]

\[ \vec{U}^{(-)}_n T + \frac{\partial T}{\partial N_T} = \vec{U}^{(-)}_n d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \]

\[ \frac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \]

\[ T_j \equiv T \text{ on } D \times (t_{j-1}, t_j). \]
Splitting method. Salinity

Step 2.

\[
S_t + (\bar{U}, \text{Grad})S - \text{Div}(\hat{a}_S \cdot \text{Grad } S) = f_S \text{ in } D \times (t_{j-1}, t_j),
\]

\[
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D,
\]

\[
\bar{U}_n^{(-)} S - \nu_S \frac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j),
\]

\[
\frac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j),
\]

\[
\bar{U}_n^{(-)} S + \frac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j),
\]

\[
\frac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j),
\]

\[
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\]
Splitting method. Circulation and sea level

\[
\begin{aligned}
\begin{cases}
\frac{\mathbf{u}^{(1)}}{t} + \begin{bmatrix} 0 & -\bar{\ell} \\ \ell & 0 \end{bmatrix} \mathbf{u}^{(1)} - g \cdot \nabla \xi = g \cdot \nabla G - \frac{1}{\rho_0} \nabla \left( P_a + g \int_0^z \rho_1(\bar{T}, \bar{S}) dz' \right) \\
\xi_t - \text{div} \left( \int_0^H \Theta\mathbf{u}^{(1)} dz \right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\
\mathbf{u}^{(1)} = \mathbf{u}_{j-1}, \ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\
\left( \int_0^H \Theta\mathbf{u}^{(1)} dz \right) \cdot n + \beta_0 m_{op} \sqrt{gH} \xi = m_{op} \sqrt{gH} d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\
\mathbf{u}^{(1)}_j \equiv \mathbf{u}^{(1)}(t_j) \text{ in } D
\end{cases}
\end{aligned}
\]

\[
\begin{aligned}
\begin{cases}
\frac{\mathbf{u}^{(2)}}{t} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \mathbf{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\
\mathbf{u}^{(2)} = \mathbf{u}^{(1)}_j \text{ при } t = t_{j-1} \text{ in } D, \\
\mathbf{u}^{(2)}_j \equiv \mathbf{u}^{(2)}(t_j) \text{ in } D,
\end{cases}
\end{aligned}
\]
The function obtained by observations processing is the function $T_{\text{obs}}$ at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. We consider this function as an approximation to SST data on $\Omega$, i.e. to $T|_{z=0}$. We assume that the function $T_{\text{obs}}$ is known only on the part of $\Omega \times (0, \bar{t})$ and we define a support of this function as $m_0$. The function of full flux $Q$ is an additional unknown function ("control") and we introduce the cost-function in the form:

$$J_\alpha(Q, T) = \frac{1}{2} \int_0^{\bar{t}} \int_\Omega \alpha |Q - Q(0)|^2 d\Omega dt + J_0(T),$$

$$J_0(T) = \frac{1}{2} \int_0^{\bar{t}} \int_\Omega m_0(T - T_{\text{obs}})R^{-1}(T - T_{\text{obs}})^T d\Omega dt.$$

Here $\alpha \equiv \alpha(\lambda, \theta, t)$ is a regularization function (it is possible that $\alpha(\lambda, \theta, t) = \text{const} \geq 0$) and it may be a dimensional quantity; $Q(0) \equiv Q(0)(\lambda, \theta, t)$ is a given function.

**Data assimilation problem.** Find the full solution $\phi$ (i.e. $T, S, \bar{u}, \xi$) of the system and function $Q$, such that, the functional $J_\alpha$ is minimal on the set of the solutions.
Optimality system

The optimality system obtained consists of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. The system of equations arising from minimization of the functional $J_\alpha$ on the set of the solution of the equations is

\[
\begin{align*}
(T_1)_t + L_1 T_1 &= F_1, \quad T_1 = T_{j-1} \quad \text{at} \quad t = t_{j-1} \\
(T_2)_t + L_2 T_2 &= F_2 + B Q_T, \quad T_2(t_{j-1}) = T_1(t_j).
\end{align*}
\]

\[
T_2(t_j) \equiv T_j \cong T \quad \text{at} \quad t = t_j.
\]

\[
\begin{align*}
(T_2^*)_t + L_2^* T_2^* &= B^* m_0 R^{-1} (T - T_{obs}), \quad T_2^*(t_j) = 0, \\
(T_1^*)_t + L_1^* T_1^* &= 0, \quad T_1^*(t_j) = T_2^*(t_{j-1}), \\
\alpha (Q - Q^{(0)}) + T_2^* &= 0.
\end{align*}
\]

Functions $T_2, Q(t_j)$ are accepted as approximations to functions $T, Q$ of the full solution for the Problem.
Optimality system in Z-coordinate and iterative process

\[
\begin{align*}
T_t + \frac{1}{2} \left( w_1 \frac{\partial T}{\partial z} + \frac{1}{r^2} \frac{\partial (r^2 w_1 T)}{\partial z} \right) - \frac{1}{r^2} \frac{\partial}{\partial z} r^2 \nu_T \frac{\partial T}{\partial z} &= f_T \\
T &= T_1(t_j) \\
-\nu_T \frac{\partial T}{\partial z} &= Q_T \text{ at } z = 0, \quad \nu_T \frac{\partial T}{\partial z} = 0 \text{ at } z = H
\end{align*}
\]

\[
\begin{align*}
-T^*_t - \frac{1}{2} \left( w_1 \frac{\partial T^*_t}{\partial z} + \frac{\partial (w_1 T^*_t)}{\partial z} \right) - \frac{1}{r^2} \frac{\partial}{\partial z} \left( r^2 \nu_T \frac{\partial T^*_t}{\partial z} \right) &= m_0 R^{-1} (T - T_{\text{obs}}) \\
T^*_t &= 0 \text{ at } t = t_j, \\
-w_1 T^*_t - \nu_T \frac{\partial T^*_t}{\partial z} &= 0 \text{ at } z = 0, \quad \nu_T \frac{\partial T^*_t}{\partial z} = 0 \text{ at } z = H,
\end{align*}
\]

\[Q^{(k+1)} = Q^{(k)} - \gamma_k (\alpha (Q^{(k)} - Q^{(0)}) + T^*) \text{ on } \Omega \times (t_0, t_1).\]
Numerical Experiments

- Numerical experiments have been carried out in the Baltic Sea (model has been developed in the INM RAS).
- Observation data: Daily sea surface temperature (SST) from Danish meteorological Institute.
- The spatial resolution is 0.0625*0.03125 degree.
- The time step is 5 minutes.
- Assimilation intervals: 4 times per day.
- The mean flux $Q^{(0)}$ was taken from the database of NCEP (National Centers for Environmental Prediction).
- Covariance matrix $R$ is calculated based on the statistical properties of observation data.

The observation data assimilation module to assimilate $T_{obs}$ was included into the Baltic Sea thermohydrodynamic model. The experiments start: (A) from 1st of January 2007; (B) from 1st June 2007. Duration of the calculation is 1 months.
Sea surface temperature (SST). Experiment (A).

(a) Observation data (average SST)  
(b) Calculation without assimilation  
(c) Calculation with assimilation
SST. Deviation from observations. Experiment (A).

(a) $T_{model} - T_{obs}$

(b) $T_{assim} - T_{obs}$
SST section. Experiment (A).

(a) Section on latitude. $58.3^\circ$ N

(b) Section on longitude. $19^\circ$ E
Salinity. Experiment (A).

(a) Model salinity

(b) Salinity after SST assimilation

(c) Section on longitude. 19° E

(d) Section on latitude. 58.3° N
Sea surface temperature. Experiment (B).

(a) Observation data (average SST) 
(b) Calculation without assimilation 
(c) Calculation with assimilation
SST. Deviation from observations. Experiment (B).

(a) $T_{\text{model}} - T_{\text{obs}}$

(b) $T_{\text{assim}} - T_{\text{obs}}$
SST section. Experiment (B).

(a) Section on latitude. $58.3^\circ$ N

(b) Section on longitude. $19^\circ$ E
Salinity. Experiment (B).

(a) Model salinity

(b) Salinity after SST assimilation

(c) Section on longitude. 19° E

(d) Section on latitude. 58.3° N
Difference in velocities

(a) Model vs Assimilation. Experiment (A)  (b) Model vs Assimilation. Experiment (B)
Summary

- The variational data assimilation problem of finding the flux on the sea surface using the observation of SST with covariance matrix in cost function was formulated and studied.

- Algorithms of the numerical solution of data assimilation problem were developed and justified. The assimilation block with covariance matrix was included into 3D hydrodynamics model developed in INM RAS.

- The numerical experiments show that assimilation of SST has a small influence to other components of the full solution, i.e. salinity, velocity etc.
References

