Precondioned GMRES Method for the Solution of Non-Symmetric Real Toeplitz Systems

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Introduction

Toeplitz Matrices

An $n \times n$ Toeplitz matrix T_n, is of the form:

$$T_n = \begin{bmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{-(n-2)} & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \cdots & t_{-(n-3)} & t_{-(n-2)} \\ t_2 & t_1 & t_0 & \cdots & t_{-(n-4)} & t_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n-2} & t_{n-3} & t_{n-4} & \cdots & t_0 & t_{-1} \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 & t_0 \end{bmatrix}.$$

We define as $T_n(f)$ the Toeplitz matrix generated by the function f.
The entries of a Toeplitz matrix $T_n(f)$ are the coefficients of the Fourier expansion of a function f:

$$t_{jk} = t_{j-k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \mathrm{e}^{-\mathrm{i}(j-k)x} \mathrm{d}x, \ \mathrm{i}^2 = -1, \ 1 \leq j, k \leq n.$$

We note that properties of the generating function f correspond to specific properties of $T_n(f)$.

f	$T_n(f)$
Real	Hermitian
Real & positive	Hermitian, positive definite & well conditioned
Real & non-negative	Hermitian, positive definite & ill conditioned
Real & even	Real & symmetric
Complex	Complex

$\kappa(A) < c$,
$c > 0$ independent of n

f	$T_n(f)$
Real	Hermitian
Real & positive	Hermitian, positive definite & well conditioned
Real & non-negative	Hermitian, positive definite & ill conditioned
Real & even	Real & symmetric
Complex	Complex

$$\boxed{\kappa(A) = \|A\|\|A^{-1}\|}$$

$\kappa(A) < c$,
$c > 0$ independent of n

f	$T_n(f)$
Real	Hermitian
Real & positive	Hermitian, positive definite & well conditioned
Real & non-negative	Hermitian, positive definite & ill conditioned
Real & even	Real & symmetric
Complex	Complex

f	$T_n(f)$
Real	Hermitian
Real & positive	Hermitian, positive definite & well conditioned
Real & non-negative	Hermitian, positive definite & ill conditioned
Real & even	Real & symmetric
Complex	Complex
Complex with Re(f): even & Im(f): odd	Real & non-symmetric

GMRES

- Krylov subspace Methods

$$K_m(A, b) = \text{span}\{b, Ab, A^2b, \ldots, A^{m-1}b\}$$

- Non-symmetric systems
- Generalized Minimal Residual Method (GMRES) (Saad and Schultz 1986)
- GMRES Converges to the solution at most in n iterations
- GMRES Converges Efficiently when $\text{Re}(\lambda_i(A)) > 0$ and $|\lambda_i(A)| < \infty$

Theoretical Background

Definition 1 (Tyrtyshnikov[1])

A set $\Phi \subset \mathbb{R}$ is called a general eigenvalue cluster for a hermitian matrix A_n, if for all $\epsilon > 0$ and Φ_ϵ being an $\epsilon-$extension of Φ:

$$\lim_{n\to\infty} \frac{\gamma_n(\epsilon)}{n} = 0,$$

where $\gamma_n(\epsilon)$ is the number of eingenvalues of A_n, that lie outside of Φ_ϵ. Φ is called a proper cluster if $\gamma_n(\epsilon) \leq c(\epsilon)$, where $c(\epsilon)$ is independent of n.

Definition 2

A set $\Phi \subset \mathbb{R}^+$ is called a general singular value cluster for a matrix A_n, if:

$$\lim_{n\to\infty} \frac{\gamma_n(\epsilon)}{n} = 0,$$

where $\gamma_n(\epsilon)$ is the number of singular values of A_n, that lie outside of Φ_ϵ. Φ is called a proper cluster if $\gamma_n(\epsilon) \leq c(\epsilon)$, where $c(\epsilon)$ is independent of n.

[1] Tyrtyshnikov, E. E. (1996). A unifying approach to some old and new theorems on distribution and clustering. Linear Algebra and its Applications, 232, 1-43.

Theorem 3 (Tilli[2])

Let $f \in \mathrm{L}^2([-\pi, \pi])$ be the generating function of $T_n(f)$. Then, the set $[\min |f|, \max |f|]$ is a general singular value cluster for $T_n(f)$.

It is proven that all the singular values are less than $\max |f|$ and at most $o(n)$ singular values of $T_n(f)$ are smaller than $\min |f|$.

[2] Tilli, P. (1998). Singular values and eigenvalues of non-Hermitian block Toeplitz matrices. Linear algebra and its applications, 272(1-3), 59-89.

Theorem 4

Let $f \in \mathrm{L}^2([-\pi, \pi])$ *and* p *be a trigonometric polynomial such that* $\mathrm{Re}\left(\frac{f}{p}\right) > 0$ *and:*

$$0 < \alpha = \min_{-\pi \leq x \leq \pi} \left| \frac{f(x)}{p(x)} \right| \leq \max_{-\pi \leq x \leq \pi} \left| \frac{f(x)}{p(x)} \right| = \beta < \infty.$$

Then the interval $[\alpha, \beta]$ *is a general cluster of the singular values of the preconditioned matrix* $T_n^{-1}(p)T_n(f)$.

We prove here that at most $o(n)$ singular values of $T_n^{-1}(p)T_n(f)$ are smaller than α and at most a constant number of singular values are greater than β.

Proof

- The singular values are the square roots of the eigenvalues of the preconditioned system of the normal equations:

$$\begin{aligned}\left(T_n^{-1}(p)\,T_n(f)\right)^H T_n^{-1}(p)\,T_n(f) &= T_n^H(f)\,T_n^{-H}(p)\,T_n^{-1}(p)\,T_n(f)\\ &= T_n(\bar{f})\,T_n^{-1}(\bar{p})\,T_n^{-1}(p)\,T_n(f)\end{aligned}$$

Proof

- The singular values are the square roots of the eigenvalues of the preconditioned system of the normal equations:

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- This matrix is similar with:

$$A_n(f,p) = T_n^{-1}(p)\,T_n(f)\,T_n(\bar{f})\,T_n^{-1}(\bar{p})$$

Proof

$$
\begin{aligned}
A_n(f,p) &= T_n^{-1}(p)\,T_n(f)\,T_n(\bar{f})\,T_n^{-1}(\bar{p}) \\
&= T_n^{-1}(p)\left(T_n(p)\,T_n\left(\frac{f}{p}\right)+E_n\right)\left(T_n\left(\frac{\bar{f}}{\bar{p}}\right)T_n(\bar{p})+E_n^H\right)T_n^{-1}(\bar{p}) \\
&= T_n\left(\frac{f}{p}\right)T_n\left(\frac{\bar{f}}{\bar{p}}\right)+R_n \leq T_n\left(\frac{f\bar{f}}{p\bar{p}}\right)+R_n = T_n\left(\left|\frac{f}{p}\right|^2\right)+R_n,
\end{aligned}
$$

where R_n is a low rank matrix, with rank at most $2d-2$, where d is the bandwidth of the band Toeplitz matrix $T_n(p)$.

From Tilli's Theorem, the general cluster is achieved.

Moreover, since of the low rank matrix R_n, at most a constant number of singular values may be greater than β.

Construction of the Preconditioner

Approximation of f

We construct the Band Toeplitz Preconditioner $T_n(p)$, ganerating by the trigonometric polynomial p, in order to obtain a cluster of the singular values of the preconditioned matrix $T_n(p)^{-1}T_n(f)$ in a small interval arroud 1, as well as to obtain a good cluster of its eigenvalues in a small bounded region of the right halfplane arround 1.
To do this, we have to choose p as a good approximation of f:

Let $f = f_1 + \mathrm{i} f_2$, where $\begin{cases} f_1\text{: even} \\ f_2\text{: odd} \end{cases}$

Let $f_1 > 0$. We can approximate f:

1. Best uniform trigonometric approximation
   - $f_1 \leftarrow$ even trigonometric polynomial of degree d_1
   - $f_2 \leftarrow$ odd trigonometric polynomial of degree d_2
   - Remez exchange algorithm
   - Nodes: k Chebyshev points of the first kind mapped on $[0, \pi]$
2. Analogous by trigonometric interpolation

The singular values of $T_n^{-1}(p)T_n(f)$ behave as the function $\left|\frac{f}{p}\right|$:

$$f_1(x) = p_1(x) + e_1(x) \text{ and } f_2(x) = p_2(x) + e_2(x),$$

where e_1 and e_2,the error functions and $\|e_1\|_\infty = \epsilon_1$ and $\|e_2\|_\infty = \epsilon_2$, the approximation errors. Then,

$$\frac{f}{p} = \frac{f_1 + \mathrm{i}f_2}{p_1 + \mathrm{i}p_2} = \frac{p_1 + e_1 + \mathrm{i}(p_2 + e_2)}{p_1 + \mathrm{i}p_2} = 1 + \frac{e_1 + \mathrm{i}e_2}{p_1 + \mathrm{i}p_2} \Rightarrow$$

$$\left|\frac{f}{p} - 1\right| = \left|\frac{e_1 + \mathrm{i}e_2}{p_1 + \mathrm{i}p_2}\right| = \left(\frac{e_1^2 + e_2^2}{p_1^2 + p_2^2}\right)^{\frac{1}{2}} \tag{1}$$

$$\leq \max_{-\pi \leq x \leq \pi} \left(\frac{2}{p_1^2(x) + p_2^2(x)}\right)^{\frac{1}{2}} \max\{\epsilon_1, \epsilon_2\} = M\epsilon.$$

The errors ϵ_1 and ϵ_2, reduce as the degrees d_1 and d_2 increase. Thus, for a prescribed $\epsilon > 0$, we can choose d_1 and d_2, such that $\max\{\epsilon_1, \epsilon_2\} \leq \epsilon$. This means that

$$\frac{f}{p} \in [1 - M\epsilon, 1 + M\epsilon]$$

.

Thus, from Tilli's Theorem, the interval $[1 - M\epsilon, 1 + M\epsilon]$ is a general singular value cluster of $Tn(p)^{-1}T_n(f)$.

Remark

The approximation above gives also a kind of cluster of the shape $\frac{f}{p}$ in a small region of the complex plane arround $\mathbf{1}$.

This results that the eigenvalues of $Tn(p)^{-1}T_n(f)$ are clustered in an analogous region, as we will see in a variate of numerical experiments.

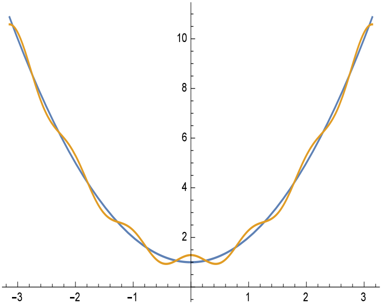
Numerical Results

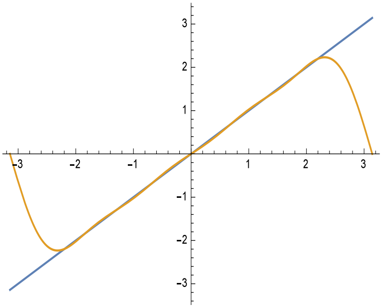
- MATLAB
- We solve $T_n(f)x = b$ by PGMRES method
- $x_0 = 0$
- Stopping criterion: $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} \leq 10^{-6}$

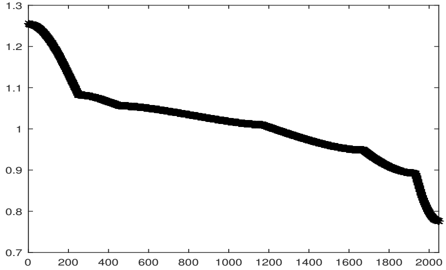
$r^{(k)}$: residual vector of k iteration and $r^{(0)} = b$

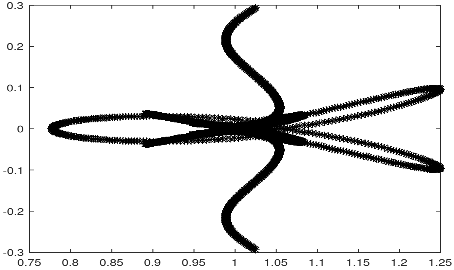
$x^2 + 1 + \mathrm{i}x$

n	I_n	$R_{4,4}$	$R_{6,4}$	$CGN_{4,4}$
256	37	8	7	11
512	36	8	7	11
1024	35	8	7	11
2048	34	8	7	11






1.3
1.2
1.1
1
0.9
0.8
0.7
0
200
400
600
800
1000
1200
1400
1600
1800
2000


0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.75
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25

Root at $x_0 = 0$

When f has roots, we have first to elimnate the roots and then to form the approximation technique, described above.

First we study the case where the root is at zero.

Let m_1, m_2 be the multiplicities of the root of f_1 and f_2, respectively. We separate the following cases:

1. $m_1 < m_2$
2. $m_1 > m_2$

Obviously, in the first case the multiplicity of the root dependson m_1, while in the second case on m_2.

1. Root at $x_0 = 0$, with $m_1 < m_2$

Following the technique proposed by Raymond Chan[3] we eliminate the root of f_1, dividing by the trigonometric polynomial :

$$g = (2 - 2\cos(x))^{\left(\frac{m_1}{2}\right)}.$$

Thus, we obtain the function

$$\hat{f} = \frac{f}{g} = \frac{f_1 + \mathrm{i} f_2}{g} = \frac{f_1}{g} + \mathrm{i}\frac{f_2}{g} = \hat{f}_1 + \mathrm{i}\hat{f}_2.$$

for which: $\hat{f}_1 > 0$.

[3] Chan, R. H. (1991). Toeplitz preconditioners for Toeplitz systems with nonnegative generating functions. IMA journal of numerical analysis, 11(3), 333-345.

Now, we form the previous technique to approximate $\hat{f}_1$ and $\hat{f}_2$ by the trigonometric polynomials q_1 and q_2, respectively.

Then, $q = q_1 + \mathrm{i}q_2$ is the associated approximation of $\hat{f}$.

Dividing by q, we get :

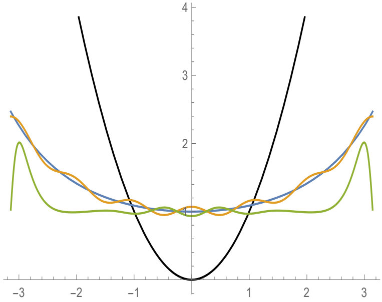
$$\frac{\hat{f}}{q} = \frac{\frac{f}{g}}{q} = \frac{\frac{f_1 + \mathrm{i}f_2}{g}}{q_1 + \mathrm{i}q_2} = \frac{f_1 + \mathrm{i}f_2}{gq_1 + \mathrm{i}gq_2} = \frac{f}{p} \tag{2}$$

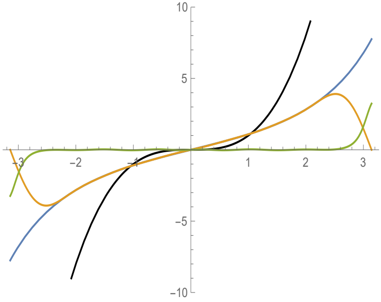
We choose the band Toeplitz matrix $T_n(p)$ as a preconditioner, where $p = gq_1 + \mathrm{i}gq_2$.

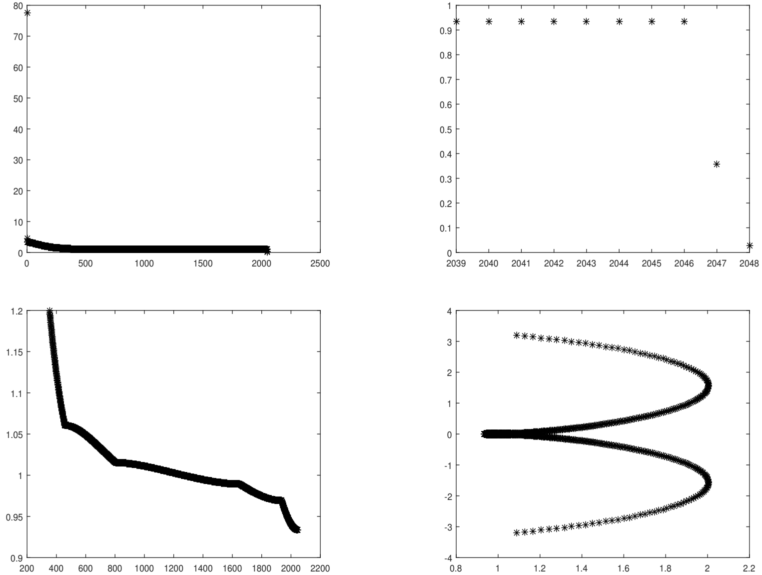
The root of f_2 is not eliminated, but it does not matter.

$f(x) = x^2 + \mathrm{i}x^3$

n	I_n	B	$R_{4,4}$	$R_{6,6}$	$CGN_{6,6}$
256	256	67	24	22	38
512	512	70	27	26	45
1024	1022	69	28	27	50
2048	2029	68	28	27	55



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60
50
40
30
20
10
0
0
500
1000
1500
2000
2500
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2039
2040
2041
2042
2043
2044
2045
2046
2047
2048
1.2
1.15
1.1
1.05
1
0.95
0.9
200
400
600
800
1000
1200
1400
1600
1800
2000
2200
4
3
2
1
0
-1
-2
-3
-4
0.8
1
1.2
1.4
1.6
1.8
2
2.2

2. Root at $x_0 = 0$, with $m_1 > m_2$

If we try to eliminate the root of f_1, using the same technique, the imaginary term will tend to infinity at zero.

If we try to eliminate the root of f_2 dividing by:

$$\mathrm{i}\,(\sin(x))^{m_2}\,,$$

then the imaginary part of f becomes the real part of $\hat{f}$ and the real part of f becomes the imaginary part of $\hat{f}$. It also holds: $\mathrm{Re}(\hat{f}) > 0$.

The problem that arises in this case is that:

$$\lim_{x \to \pm\pi} \mathrm{Re}(\hat{f}) = +\infty.$$

To avoid this problem we divide by the sum of both functions that eliminate the roots:

$$g = g_1 + \mathrm{i}g_2 = (2 - 2\cos(x))^{\frac{m_1}{2}} + \mathrm{i}(\sin(x))^{m_2}$$

Then we get the function:

$$\hat{f} = \frac{f_1 + \mathrm{i}f_2}{g_1 + \mathrm{i}g_2} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} + \mathrm{i}\frac{f_2 g_1 - f_1 g_2}{g_1^2 + g_2^2} = \hat{f}_1 + \mathrm{i}\hat{f}_2.$$

To avoid this problem we divide by the sum of both functions that eliminate the roots:

$$g = g_1 + \mathrm{i} g_2 = (2 - 2\cos(x))^{\frac{m_1}{2}} + \mathrm{i}\,(\sin(x))^{m_2}$$

Then we get the function:

$$\hat{f} = \frac{f_1 + \mathrm{i} f_2}{g_1 + \mathrm{i} g_2} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} + \mathrm{i}\frac{f_2 g_1 - f_1 g_2}{g_1^2 + g_2^2} = \hat{f}_1 + \mathrm{i}\hat{f}_2.$$

- $\hat{f}_1 > 0$
- $|\hat{f}| < \infty$
- $\hat{f}_2(0) = 0$

To avoid this problem we divide by the sum of both functions that eliminate the roots:

$$g = g_1 + \mathrm{i}g_2 = (2 - 2\cos(x))^{\frac{m_1}{2}} + \mathrm{i}(\sin(x))^{m_2}$$

Then we get the function:

$$\hat{f} = \frac{f_1 + \mathrm{i}f_2}{g_1 + \mathrm{i}g_2} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} + \mathrm{i}\frac{f_2 g_1 - f_1 g_2}{g_1^2 + g_2^2} = \hat{f}_1 + \mathrm{i}\hat{f}_2.$$

- $\hat{f}_1 > 0$
- $|\hat{f}| < \infty$
- $\hat{f}_2(0) = 0$

Now we are allowed to approximate the function $\hat{f}$.

Let $q = q_1 + \mathrm{i}q_2$ be the approximation of $\hat{f}$. Dividing by q we have:

$$\frac{\hat{f}}{q} = \frac{\frac{f}{g}}{q} = \frac{f}{gq} = \frac{f}{p},$$

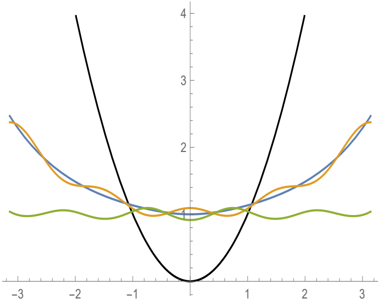
where:

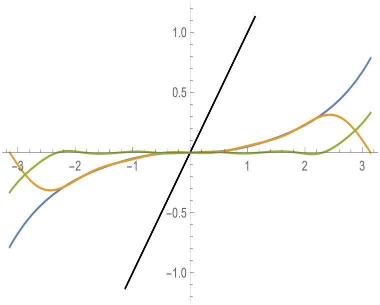
$$p = gq = (g_1 + \mathrm{i}g_2)(q_1 + \mathrm{i}q_2) = g_1q_1 - g_2q_2 + \mathrm{i}(g_1q_2 + g_2q_1) = p_1 + \mathrm{i}p_2.$$

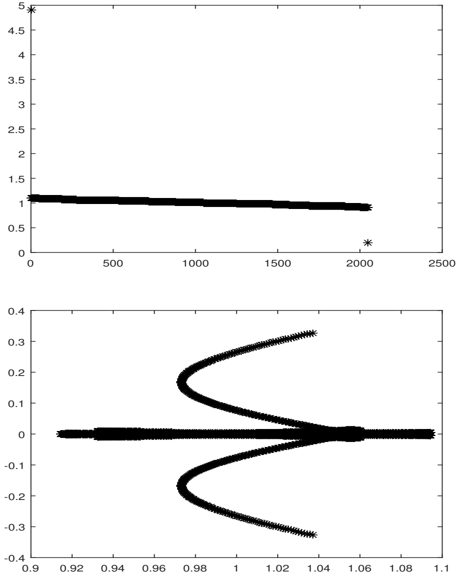
Choosing the band Toeplitz matrix $T_n(p)$ as a preconditioner we achieve a singular value cluster.

$f(x) = x^2 + \mathrm{i}x$

n	I_n	B	$R_{4,4}$	$CGN_{4,4}$
256	256	11	6	8
512	512	11	6	9
1024	1024	10	6	9
2048	2048	10	6	10






5
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3.5
3
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1.5
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0.5
0
0
500
1000
1500
2000
2500
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0.9
0.92
0.94
0.96
0.98
1
1.02
1.04
1.06
1.08
1.1

$f(x_0) = 0$, with $x_0 \neq 0$

Suppose that f has a root at the point $x_0 \in [-\pi, \pi]$, $x_0 \neq 0$. Thus:

- $f_1(x_0) = 0$
- $f_2(x_0) = 0$

Let m_1 and m_2, are the multiplicities of the root of f_1 and f_2, respectively.

Since, f_1: even and f_2: odd, $-x_0$ is also a root with the same multiplicities

f_2 has an additional root at 0 with odd multiplicity m_0.

The function f_1 should have the form:

$$f_1(x) = c_1(x)(x - x_0)^{m_1}(x + x_0)^{m_1} \quad (3)$$

in small regions of $\pm x_0$:

$$I_\epsilon = [-\epsilon - x_0, -x_0 + \epsilon] \cup [x_0 - \epsilon, x_0 + \epsilon],$$

where c_1 is a bounded function far from zero and conserves the sign in I_ϵ. We eliminate them dividing by:

$$\begin{aligned} &\operatorname{sign}(c_1(x)) \left(\sin\left(\frac{x - x_0}{2}\right)\right)^{m_1} \left(\sin\left(\frac{x + x_0}{2}\right)\right)^{m_1} \\ &= \operatorname{sign}(c_1(x)) \frac{1}{2^{\frac{m_1}{2}}} \left(\cos(x_0) - \cos(x)\right)^{m_1}, \end{aligned}$$

or by the simplest form:

$$g_1(x) = \operatorname{sign}(c_1(x)) \left(\cos(x_0) - \cos(x)\right)^{m_1} \quad (4)$$

Following the same analysis for the function f_2, its roots are eliminated by:

$$g_2(x) = \text{sign}(c_2(x))\,(\cos(x_0) - \cos(x))^{m_2}\,(\sin(x))^{m_0}\,, \tag{5}$$

where c_2 plays the same role as c_1.
We have the following cases:

1. $m_1 \leq m_2$
   We need to divide by:
   $$g = g_1 = \text{sign}(c_1(x))\,(\cos(x_0) - \cos(x))^{m_1}\,.$$
2. $m_1 > m_2$
   We need to divide by the sum:
   $$\begin{aligned} g = g_1 + \mathrm{i} g_2 = {} & \text{sign}(c_1(x))\,(\cos(x_0) - \cos(x))^{m_1} \\ & + \mathrm{i}\,\text{sign}(c_2(x))\,(\cos(x_0) - \cos(x))^{m_2}\,(\sin(x))^{m_0}\,. \end{aligned} \tag{6}$$

Following the same analysis for the function f_2, its roots are eliminated by:

$$g_2(x) = \text{sign}(c_2(x))\left(\cos(x_0) - \cos(x)\right)^{m_2}\left(\sin(x)\right)^{m_0}, \tag{5}$$

where c_2 plays the same role as c_1.

We have the following cases:

1. $m_1 \leq m_2$

   We need to divide by:

   $$g = g_1 = \text{sign}(c_1(x))\left(\cos(x_0) - \cos(x)\right)^{m_1}.$$

2. $m_1 > m_2$

   We need to divide by the sum:

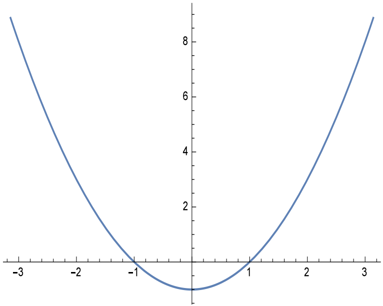
   $$\begin{aligned} g = g_1 + \mathrm{i} g_2 = {} & \text{sign}(c_1(x))\left(\cos(x_0) - \cos(x)\right)^{m_1} \\ & + \mathrm{i}\,\text{sign}(c_2(x))\left(\cos(x_0) - \cos(x)\right)^{m_2}\left(\sin(x)\right)^{m_0}. \end{aligned} \tag{6}$$

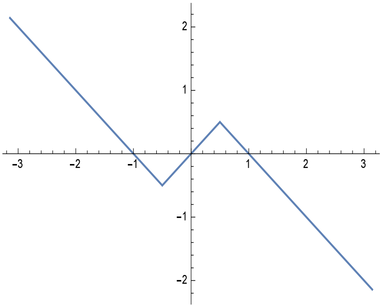
Then we follow the same technique to approximate $\hat{f} = \frac{f}{g}$ by a trigonometric polynomial and finally, the band Toeplitz preconditioner $T_n(p)$ will be generated by $p = gq$.

Example with $m_1 = m_2$

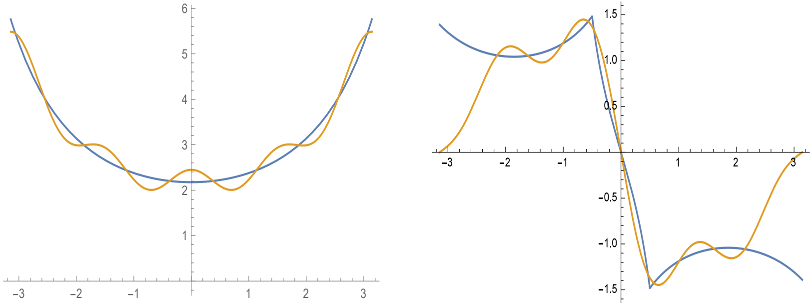
Let $f = f_1 + \mathrm{i} f_2$, where:

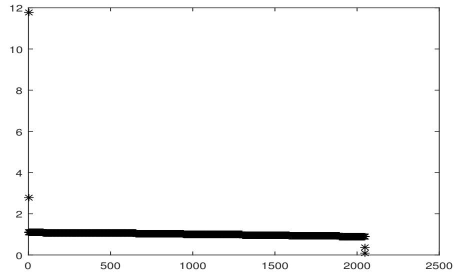
$$f_1(x) = x^2 - 1 \text{ and } f_2(x) = \begin{cases} -1 - x \,, & -\pi \le x \le -\frac{1}{2} \\ x \,, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 1 - x \,, & \frac{1}{2} \le x \le \pi \end{cases} \tag{7}$$

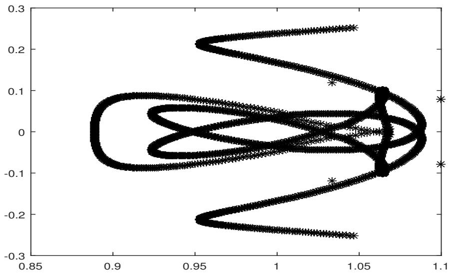




n	I_n	B	$R_{4,4}$	$CGN_{4,4}$
256	256	15	6	11
512	512	15	6	12
1024	1024	16	6	13
2048	2047	15	6	15




12
10
8
6
4
2
0
0
500
1000
1500
2000
2500


0.3
0.2
0.1
0
-0.1
-0.2
-0.3
0.85
0.9
0.95
1
1.05
1.1

Pathological case

- $f_1(\pm x_1) = 0$, with multiplicity m_1
- $f_2(\pm x_2) = 0$, with multiplicity m_2
- $f_2(0) = 0$, with miltiplicity m_0

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In this case we could not find a trigonometric polynomial g, such that:

$$\mathrm{Re}\left(\frac{f}{g}\right) > 0 \quad \text{and} \quad 0 < \left|\frac{f}{g}\right| < \infty.$$

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In this case we could not find a trigonometric polynomial g, such that:

$$\mathrm{Re}\left(\frac{f}{g}\right) > 0 \text{ and } 0 < \left|\frac{f}{g}\right| < \infty.$$

This case will probably be covered by a different technique.

Roots at multiple points

Let $f_1(\pm x_i) = f_2(\pm x_i) = 0$, $i = 1, 2, \ldots, k$ with multiplicities m_i and ℓ_i, respectively and $f_2(0) = 0$, with multiplicity ℓ_0.

- $m_i \leq \ell_i$, $\forall i = 1, 2, \ldots, k$

$$g = \prod_{i=1}^{k} \mathrm{sign}(c_i(x)) \left(\cos(x_i) - \cos(x)\right)^{m_i}$$

- $m_i > \ell_i$, $\forall i = 1, 2, \ldots, k$

$$g = \prod_{i=1}^{k} \mathrm{sign}(c_{1,i}(x)) \left(\cos(x_i) - \cos(x)\right)^{m_i}$$

$$+\, \mathrm{i} \left(\sin(x)\right)^{\ell_0} \prod_{i=1}^{k} \mathrm{sign}(c_{2,i}(x)) \left(\cos(x_i) - \cos(x)\right)^{\ell_i}$$

- $\exists i, j$: $m_i \leq \ell_i$ and $m_j > \ell_j$ $\rightarrow$ Pathological case

Extension to two-level nonsymmetric Toeplitz systems

This preconditioning technique can be extended to two-level nonsymmetric Toeplitz systems.

These matrices are Block Toeplitz with Toeplitz Blocks (BTTB) matrices and are generated by two variate functions:

$$f(x,y) = f_1(x,y) + \mathrm{i} f_2(x,y), \quad (x,y) \in [-\pi,\pi]^2,$$

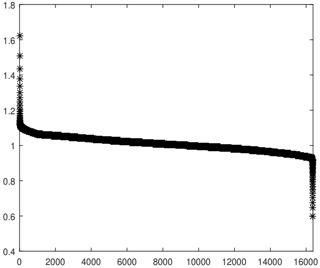
with f_1: even and f_2: odd, in both variables.

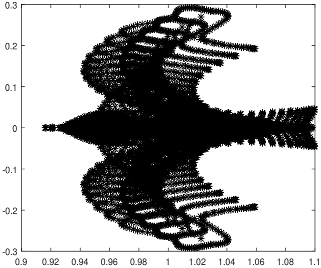
Difficulties:

- More Complicated Computations.
- There not exists Best Uniform Approximation. Instead, Trigonometric Polynomial Interpolation in two variables, is used.

$f(x,y) = x^2 + y^2 + \mathrm{i}(x+y)$

$n \times m$	$R_{4,4}$
16×16	7
32×32	7
64×64	8
128×128	8





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Thank you

for your attention