# Quasi-Toeplitz matrices: analysis, algorithms and applications

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Joint work with

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# Outline

Toeplitz matrices

Quasi-Toeplitz (QT) matrices

Quadratic matrix equations with QT coefficients

Functions of a QT matrix

# Toeplitz matrices

A matrix $T = (t_{i,j})$ is Toeplitz if there exists a sequence $\{a_k\}_{k\in\mathbb{Z}}$ such that $t_{i,j} = a_{j-i}$

A Toeplitz matrix $T$ can be

- finite $n \times n$ if $i, j = 1, \ldots, n$
- semi-infinite if $i, j \in \mathbb{Z}^+$
- bi-infinite if $i, j \in \mathbb{Z}$

For instance,

$$\begin{bmatrix} a_0 & a_1 & & & \\ a_{-1} & a_0 & a_1 & & \\ & a_{-1} & a_0 & a_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & a_{-1} & a_0 & a_1 & & \\ & & a_{-1} & a_0 & a_1 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

# Toeplitz matrices

- An infinite Toeplitz matrix, or a sequence of $n \times n$ Toeplitz matrices, can be associated with a formal Laurent series $a(z) = \sum_{k=-\infty}^{+\infty} a_k z^k$
- In this case we write $T = T(a)$
- Observe that if $a(z)$ is a Laurent polynomial, say, $a(z) = \sum_{k=-1}^{2} a_k z^k$ then $T(a)$ is banded

$$T(a) = \begin{bmatrix} a_0 & a_1 & a_2 & & \\ a_{-1} & a_0 & a_1 & a_2 & \\ & a_{-1} & a_0 & a_1 & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

# Toeplitz matrices in the applications

Toeplitz matrices are encountered in the mathematical models where some entity enjoys some shift invariance property

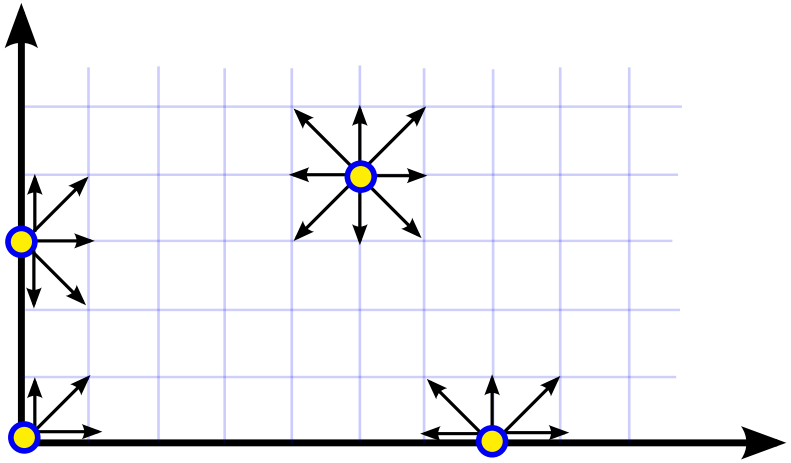
For this reason, they are ubiquitous

Typical examples

- FIR filters: the impulse response is shift invariant
- Integral equations $\int_a^b k(x-t)f(t)dt = g(x)$
- Image restoration with shift invariant PSF
- Queueing models where probability is shift-invariant
- Finite differences discretization of derivatives, say $-f''(x)$ and $\mathrm{trid}(-1, 2, -1)$

# Problem 1: a bi-dimensional random walk

- In the quarter plane $\{(i,j),\ i,j \in \mathbb{N}_+\}$, a particle can remain in the same position or move to the 8 neighbouring positions with given probabilities



- These probabilities are independent of the position of the particle
- When the particle reaches the boundaries $i = 0$ or $j = 0$, it is reflected with given probabilities

**Applications:** Tandem queues with infinite buffer

# A bi-dimensional random walk (cont.)

Let $X_n$ be the position of the particle at time $n$.
Then $\{X_n\}_n$ is a Markov chain with transition probability matrix

$$\mathcal{P} = \begin{bmatrix} B_0 & B_1 & & & \\ A_{-1} & A_0 & A_1 & & \\ & A_{-1} & A_0 & A_1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$B_i = \begin{bmatrix} \hat{b}_{0,i} & \hat{b}_{1,i} & & \\ b_{-1,i} & b_{0,i} & b_{1,i} & \\ & \ddots & \ddots & \ddots \end{bmatrix}, \quad A_i = \begin{bmatrix} \hat{a}_{0,i} & \hat{a}_{1,i} & & \\ a_{-1,i} & a_{0,i} & a_{1,i} & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

and $B_0 + B_1$, $A_{-1} + A_0 + A_1$ row stochastic.

**Structure:** semi-infinite block tridiagonal, quasi block Toeplitz matrix with semi-infinite tridiagonal, quasi Toeplitz blocks

# Matrix geometric approach

Neuts 95, Latouche Ramaswami 98

**Problem:** Compute, in the positive recurrent case, the invariant probability vector $\pi$, such that $\pi^T \mathcal{P} = \pi^T$ and $\|\pi\|_1 = 1$.

- Partition $\pi$ into blocks $\pi_i$, $\pi^T = [\pi_0^T,\ \pi_1^T,\ \pi_2^T, \ldots]$
- The vector $\pi$ is such that $\pi_i^T = \pi_0^T R^i$, $i = 1, 2, \ldots$, where $R$ is the minimal nonnegative solution of the matrix equation

$$X = X^2 A_{-1} + X A_0 + A_1$$

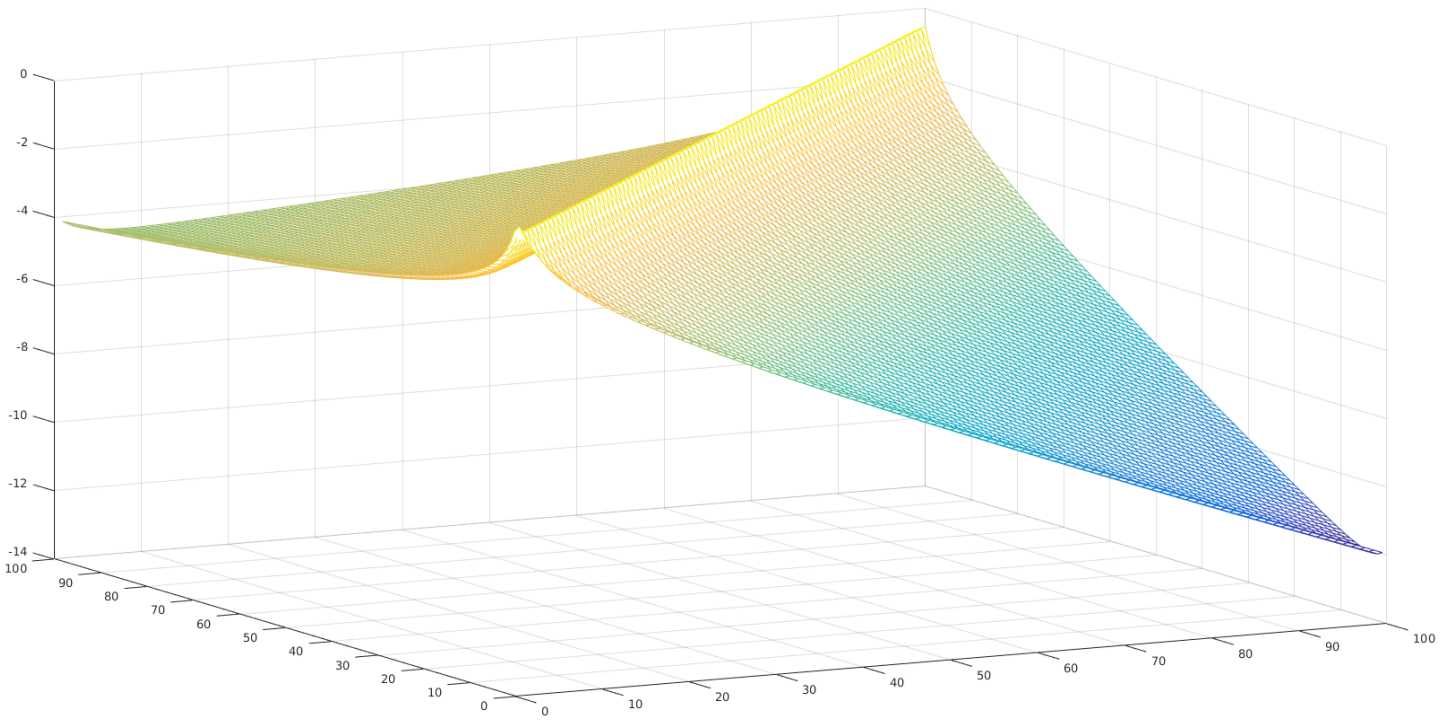
- $\pi_0$ is such that $\pi_0^T (B_0 + B_1 G) = 0$ where $G$ is the minimal nonnegative solution of the matrix equation

$$X = A_{-1} + A_0 X + A_1 X^2$$

# Questions

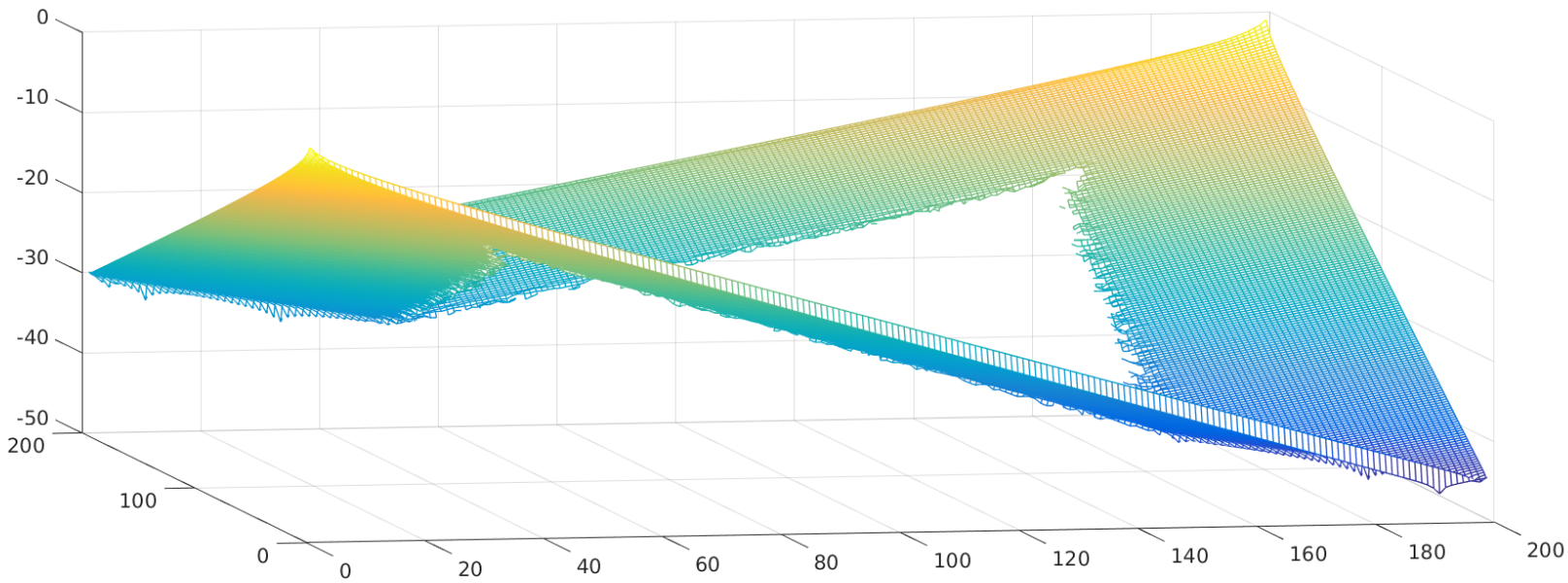
- How to solve these equations with infinite-dimensional block coefficients?
- Can we exploit the structure of the matrix coefficients?
- Do the solutions $G$ and $R$ have a (Toeplitz?) structure?

**Numerical evidence**: Log plot of a portion of the solution $G$



# Numerical evidence

Log plot of the displacement shift
$\Delta = G(1:n-1, 1:n-1) - G(2:n, 2:n)$




Numerically, the displacement shift of $G$ goes exponentially to zero moving from the diagonal corners to the center.

# Problem 2: Numerical solution of the heat equation

Solve

$$\frac{\partial u(x,t)}{\partial t} = \gamma \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \quad \gamma > 0$$

with bound./init. conditions $u(0,t) = u(1,t) = 0$, $u(x,0) = f(x)$.

The discretization of the second derivative by means of finite differences with the grid $x_i = ih$, $i = 0, 1, \ldots, n+1$, with $h = 1/(n+1)$, leads to the IVP

$$\begin{aligned} v'(t) &= \gamma A v(t), \\ v(0) &= g \end{aligned}$$

where $v(t) = (v_i(t))$, $v_i(t) = u(x_i, t)$, $g = (g_i)$, $g_i = f(x_i)$,

$$A = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}$$

# Matrix exponential

- A solution to this IVP is given by

$$v(t) = \exp(\gamma t A) g$$

so that, setting $t_j = j\Delta_t$ for a given time step $\Delta_t$, we have

$$v(t_j) = \exp(\gamma \Delta_t A) v(t_{j-1})$$

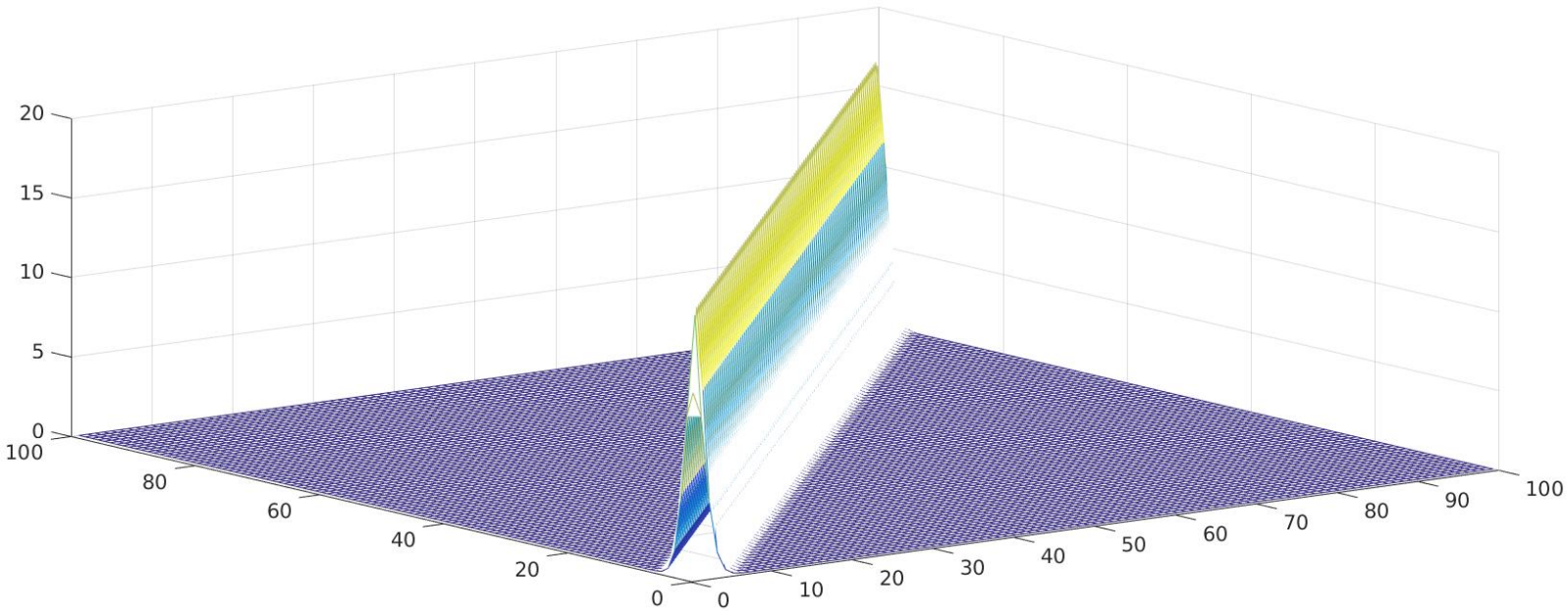
- Finite spatial domain, say $[0, 1]$, leads to a matrix exponential of a finite Toeplitz matrix
- Semi-infinite domain, say $[0, \infty)$, leads to a matrix exponential of a semi infinite Toeplitz matrix

**Applications:** mathematical models from finance, Black and Scholes equation, Merton integral differential equation [Merton 1976], [Kressner, Luce 2018]

# Questions

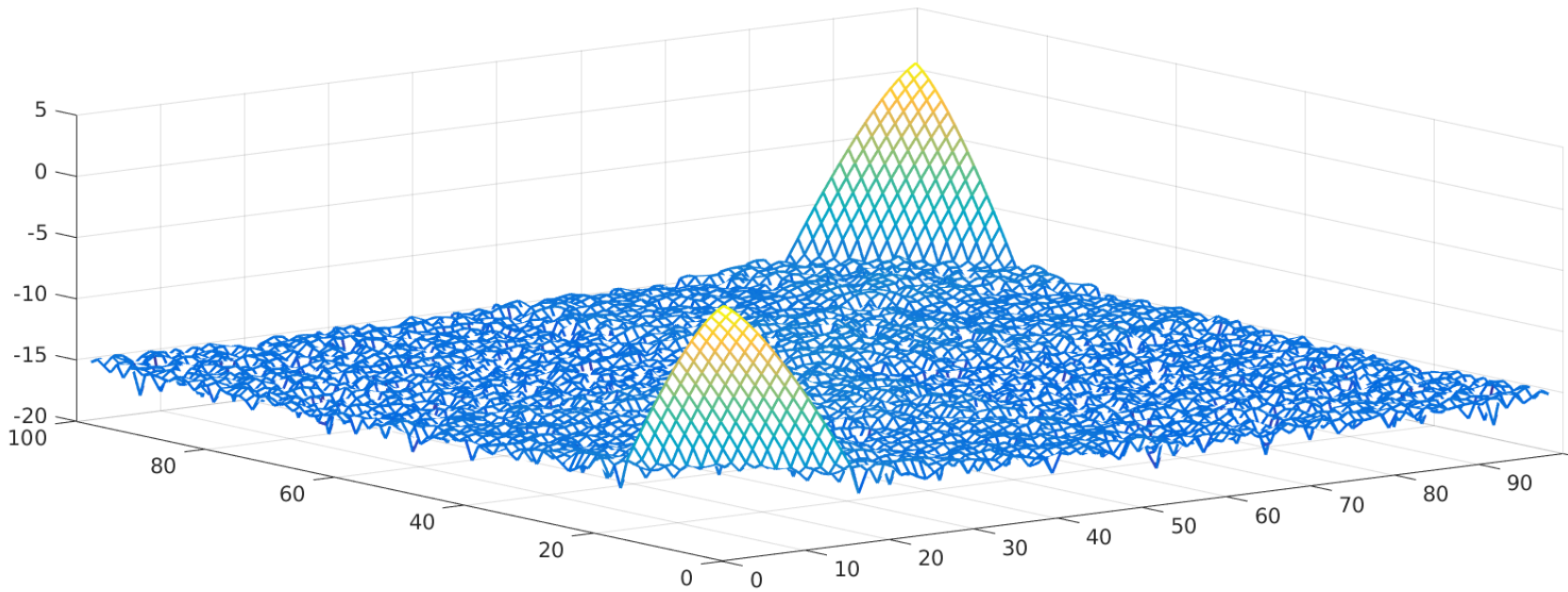
- If $T = T(a)$ is a Toeplitz matrix, is $\exp(T)$ Toeplitz as well? How "far" is $\exp(T)$ from $T(\exp(a))$?
- In general, given $f : \mathbb{C} \to \mathbb{C}$ and a Toeplitz matrix $T = T(a)$, is $f(T)$ Toeplitz as well? How "far" is $f(T)$ from $T(f(a))$?

**Numerical evidence**: Take $T = \mathrm{trid}(1, 2, 1)$ of size $n = 100$, compute $B = \exp(T)$ with the function `expm` of Matlab and plot it



# Numerical Evidence

Log plot of $|\Delta| = |B(1:n-1, 1:n-1) - B(2:n, 2:n)|$




Numerically the exponential is still Toeplitz except for the few entries in two opposite corners

# Problem analysis

1. We introduce the class of Quasi-Toeplitz matrices, consisting of matrices of the form $T(a) + E$, where $T(a)$ is the Toeplitz part and $E$ is a *"compact correction"*
2. We introduce a matrix arithmetic and a Banach algebra containing these matrices
3. Matrices belonging to this algebra can be represented by means of a finite number of parameters, with a given representation error
4. We show that, under suitable assumptions, the sought solutions $G$ and $R$ of the quadratic equations, as well as the matrix function $f(T)$, with $T$ Toeplitz, are Quasi-Toeplitz matrices

# Some definitions

- **Unit circle**: $\mathbb{T} = \{z \in \mathbb{C} : \quad |z| = 1\}$
- **Wiener class**:
  $\mathcal{W} = \{a(z) = \sum_{i\in\mathbb{Z}} a_i z^i : \mathbb{T} \to \mathbb{C}, \quad \sum_{i\in\mathbb{Z}} |a_i| < \infty\}$
  $\|a\|_{\mathcal{W}} := \sum_{i\in\mathbb{Z}} |a_i|$
- **Subset of $\mathcal{W}$**:
  $\mathcal{W}_1 = \{a(z) \in \mathcal{W} : \quad a'(z) \in \mathcal{W}\}$ where $a'(z) = \sum_{i\in\mathbb{Z}} i a_i z^{i-1}$
  $\|a\|_{\mathcal{W}_1} := \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}}$
- **Semi-infinite Toeplitz matrix associated with $a(z) \in \mathcal{W}$**:
  $T(a) = (t_{i,j}),\ t_{i,j} = a_{j-i},\ i, j \in \mathbb{Z}^+$

# Some properties

Boettcher, Grudsky, Silbermann

- $\mathcal{W}$ with the norm $\|a\|_{\mathcal{W}}$ is a Banach algebra, that is,
  - a linear space
  - closed under multiplication
  - the norm is sub-multiplicative, i.e., $\|ab\|_{\mathcal{W}} \leq \|a\|_{\mathcal{W}} \|b\|_{\mathcal{W}}$
  - complete space, i.e., any Cauchy sequence converges in $\mathcal{W}$
- $\mathcal{W}_1$ with the norm $\|a\|_{\mathcal{W}_1}$ is a Banach algebra
- $\|T(a)\|_p \leq \|a\|_{\mathcal{W}}$, for any $p \geq 1$, included $p = \infty$

$$\|A\|_p = \sup_{\|x\|_p=1} \|Ax\|_p, \quad \|x\|_p = \left( \sum_{i \in \mathbb{Z}^+} |x_i|^p \right)^{1/p}$$

# Product of Toeplitz matrices

Widom, Boettcher, Grudsky, Silbermann

**Theorem**

*For $a(z), b(z) \in \mathcal{W}$, let $c(z) := a(z)b(z)$, then*

$$T(a)T(b) = T(c) - H(a_-)H(b_+),$$

*where $H(a_-) = (a_{-i-j+1})_{i,j\in\mathbb{Z}^+}$, $H(b_+) = (b_{i+j-1})_{i,j\in\mathbb{Z}^+}$ are Hankel matrices*

$$H(a_-) = \begin{bmatrix} a_{-1} & a_{-2} & a_{-3} & \dots \\ a_{-2} & a_{-3} & \iddots & \iddots \\ a_{-3} & \iddots & \iddots & \iddots \\ \vdots & \iddots & \iddots & \iddots \end{bmatrix}, \quad H(b_+) = \begin{bmatrix} b_1 & b_2 & b_3 & \dots \\ b_2 & b_3 & \iddots & \iddots \\ b_3 & \iddots & \iddots & \iddots \\ \vdots & \iddots & \iddots & \iddots \end{bmatrix}$$

**In words**: The product of two infinite Toeplitz matrices differs from a Toeplitz matrix by a correction which is located mostly in the upper left corner

# A Banach algebra

The set of matrices

$\{T(a)+K : a(z)$ is continuous in $\mathbb{T}, K$ is a compact operator in $\ell^2\}$

is a Banach algebra of operators in $\ell^2$ (Boettcher, Silbermann)

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**But** :

- $\|\cdot\|_2$ cannot be easily computed
- we would like a norm such that the Toeplitz part and the correction are separated
- in our applications some matrices are not in $\ell^2$

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**But** :

- $\|\cdot\|_2$ cannot be easily computed
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- in our applications some matrices are not in $\ell^2$

**Therefore** : we introduce a different Banach algebra

# Quasi-Toeplitz (QT) matrices

- For $E = (e_{i,j})_{i,j\in\mathbb{Z}^+}$ define $\|E\|_{\mathcal{F}} := \sum_{i,j\in\mathbb{Z}^+} |e_{i,j}|$, and

$$\mathcal{F} = \{F = (f_{i,j})_{i,j\in\mathbb{Z}^+} : \quad \|F\|_{\mathcal{F}} < +\infty\}$$

- Define **Quasi-Toeplitz (QT)** a matrix of the form

$$A = T(a) + E, \quad a(z) \in \mathcal{W}_1, \quad E \in \mathcal{F}$$

- The set $\mathcal{QT}$ of QT matrices is a Banach algebra with the norm

$$\|A\|_{\mathcal{QT}} = \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}} + \|E\|_{\mathcal{F}}$$

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- The set $\mathcal{QT}$ of QT matrices is a Banach algebra with the norm

$$\|A\|_{\mathcal{QT}} = \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}} + \|E\|_{\mathcal{F}}$$

**Pros and cons:**

☺ Computing $\|A\|_{\mathcal{QT}}$ is easy

☹ The norm is too restrictive, there are functions $a(z) \in \mathcal{W}$ such that $a'(z) \notin \mathcal{W}$

# A wider Banach algebra

For $A = T(a) + E$, with $a(z) \in \mathcal{W}$ and $E$ compact operator in $L^p$ define

$$\|A\|_{\mathcal{QT}_p} = \alpha \|a\|_{\mathcal{W}} + \|E\|_p, \quad \alpha = (1 + \sqrt{5})/2,$$

**Theorem**

*The space $\mathcal{QT}_p$ formed by matrices $A$ such that $\|A\|_{\mathcal{QT}_p} < \infty$ is a Banach algebra, moreover,*

$$\mathcal{QT} \subset \mathcal{QT}_p$$

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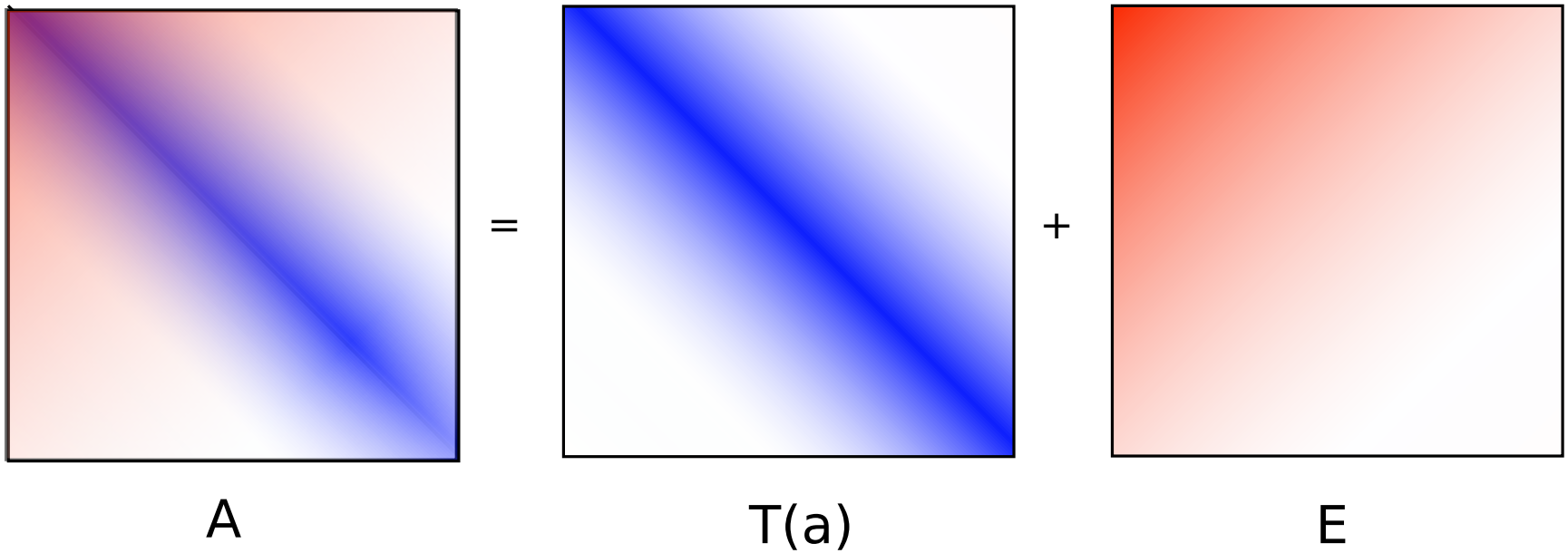
**Pros and cons:**

- ☺ the class is wider than before
- ☺☹ if $p \notin \{1, \infty\}$, $E$ can be approximated by a matrix with a finite support. This is false if $p \in \{1, \infty\}$
- ☺ if $p = 2$, SVD can be used to manipulate and compress $E$

# Pictorial representation of QT matrices




$A$ = Toeplitz part + correction

# Some computational consequences

Any matrix $A = T(a) + E$ belonging to $\mathcal{QT}$ or to $\mathcal{QT}_p$, with $1 < p < \infty$, can be represented by means of a finite set of parameters to any desired precision:

- Since $a(z)$ is represented by a bi-infinite sequence $\{a_k\}_{k\in\mathbb{Z}}$ having decay of the coefficients, we may represent $a(z)$ with a finite sequence $\mathbf{a} = (a_{n_-}, \ldots, a_0, \ldots, a_{n_+})$ up to an arbitrarily small error
- The matrix $E$ is represented by means of the pair $(U, V)$ such that $E = UV^T$, $U$ and $V$ have a finite number of columns given by the numerical rank of $E$.
- The matrices $U$ and $V$ can be truncated to a finite number of rows

# Matrix arithmetic

- Relying on this representation, a matrix arithmetic can be defined and implemented for QT matrices.

$$
\begin{aligned}
(T(a) + E_a) + (T(b) + E_b) &= T(a + b) + E_{ab} \\
(T(a) + E_a)(T(b) + E_b) &= T(ab) + F_{ab} \\
(T(a) + E_a)^{-1} &= T(a^{-1}) + G_{ab}
\end{aligned}
$$

- The inversion is based on the Wiener-Hopf factorization $a(z) = u(z)\ell(z^{-1})$
- An SVD-based compression technique is introduced in order to keep small the numerical rank of the correction matrix

# Matlab Toolbox

- The QT matrix arithmetic has been implemented in a Matlab Toolbox by Leonardo Robol and Stefano Massei https://github.com/numpi/cqt-toolbox
- The operation are invoked with the standard Matlab syntax, say,

$$\mathtt{C} = \mathtt{A} + \mathtt{B}, \quad \mathtt{C} = \mathtt{A} * \mathtt{B}, \quad \mathtt{C} = \mathtt{A} \backslash \mathtt{B}, \quad \mathtt{C} = \mathtt{A} / \mathtt{B}$$

- Other functions are available like

$$[\mathtt{U}, \mathtt{L}] = \mathtt{ul}(\mathtt{A}), \ \mathtt{B} = \mathtt{inv}(\mathtt{A}), \ \mathtt{A} = \mathtt{qt}(\mathtt{am}, \mathtt{ap}, \mathtt{E})$$

# Back to quadratic matrix equations with QT coefficients

We define Quasi Toeplitz (QT) any matrix belonging to $\mathcal{QT}$ or to $\mathcal{QT}_p$

**Question:** If the coefficients of a quadratic matrix equation $A_{-1} + A_0 X + A_1 X^2 = X$ are QT matrices, is the solution a QT matrix?

# A general result

Let $\mathcal{B}$ be a Banach algebra and $A_i \in \mathcal{B}$ for $i = -1, 0, 1$.

**Theorem**

*Assume that $\varphi(z) = z^{-1}A_{-1} + (A_0 - I) + zA_1$ is invertible for $t^{-1} < |z| < t$, with $t > 1$, and let $\psi(z) = \varphi(z)^{-1} = \sum_{i\in\mathbb{Z}} z^i H_i$. If $H_0$ is invertible then $G = H_{-1}H_0^{-1}$ and $R = H_0^{-1}H_1$ solve*

$$A_{-1} + A_0X + A_1X^2 = X \quad \text{and} \quad X^2A_{-1} + XA_0 + A_1 = X$$

*respectively, they belong to $\mathcal{B}$ and their spectrum lies inside the open unit disk. Moreover*

$$\varphi(z) = (I - zR)W(I - z^{-1}G)$$

*with $W \in \mathcal{B}$ invertible.*

**Remark:** In our framework, $\mathcal{B}$ is the Banach algebra of QT matrices, hence $G = T(g) + E_g$ and $R = T(r) + E_r$

# Solving the quadratic matrix equations

- **Idea:** Use the QT-arithmetic to extend standard algorithms, used in the finite dimensional case, to the case of QT matrix coefficients
- We provide algorithms for computing $G$ and $R$, more precisely, for computing the coefficients of $g(z)$, $r(z)$ together with "slim" matrices $U, V, Y, W$ such that

$$E_g = UV^T, \quad E_r = YW^T$$

# Fixed point iteration

- Represent $A_i = T(a_i) + E_i$, $i = -1, 0, 1$
- Set $G_0 = T(1)$ (e.g., $G_0$ is the identity matrix)
- For $k = 1, 2, \ldots$ compute $G_k = T(g_k) + E_k^{(g)}$ by means of

$$G_k = A_{-1} + A_0 G_{k-1} + A_1 G_{k-1}^2$$

  until convergence

**Remark:** Linear convergence, sometimes very slow

# Cyclic reduction

- Compute

$$A_0^{(h+1)} = A_0^{(h)} + A_1^{(h)} S^{(h)} A_{-1}^{(h)} + A_{-1}^{(h)} S^{(h)} A_1^{(h)}, \quad S^{(h)} = (I - A_0^{(h)})^{-1}$$
$$A_1^{(h+1)} = A_1^{(h)} S^{(h)} A_1^{(h)}, \quad A_{-1}^{(h+1)} = A_{-1}^{(h)} S^{(h)} A_{-1}^{(h)},$$
$$\widehat{A}^{(h+1)} = \widehat{A}^{(h)} + A_{-1}^{(h)} S^{(h)} A_1^{(h)},$$
$$\widetilde{A}^{(h+1)} = \widetilde{A}^{(h)} + A_1^{(h)} S^{(h)} A_{-1}^{(h)}, \qquad h = 0, 1, \ldots$$

with $A_0^{(0)} = \widetilde{A}^{(0)} = \widetilde{A}^{(0)} = A_0$, $A_1^{(0)} = A_1$, $A_{-1}^{(0)} = A_{-1}$, where $A_i^{(h)} = T(a_i^{(h)}) + E_i^{(h)}$, for $i = -1, 0, 1$.

- Set $G_h = (I - \widehat{A}^{(h)})^{-1} A_{-1}$, $R_h = A_1 (I - \widetilde{A}^{(h)})^{-1}$

**Remark:** Relying on the QT matrix arithmetic, cyclic reduction can be easily implemented. Under mild conditions the sequences $\{G_h\}$ and $\{R_h\}$ converge quadratically to $G$ and $R$, respectively

# Some numerical tests with CR

Ten instances of the two-node Jackson network from [Motyer and Taylor 2006]: problems 1–10

$$A_{-1} = \begin{bmatrix} (1-q)\mu_2 & q\mu_2 & & \\ & (1-q)\mu_2 & q\mu_2 & \\ & & \ddots & \ddots \end{bmatrix}$$

$$A_0 = \begin{bmatrix} -(\lambda_1+\lambda_2+\mu_2) & \lambda_1 & & \\ (1-p)\mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \lambda_2 & & \\ p\mu_1 & \lambda_2 & \\ & \ddots & \ddots \end{bmatrix}$$

# Some numerical tests

| Problem | CPU | Res | Band | Rows | Columns | Rank |
|---|---|---|---|---|---|---|
| 1 | 2.61 | $8.6 \cdot 10^{-16}$ | 561 | 541 | 138 | 8 |
| 2 | 2.91 | $1.5 \cdot 10^{-15}$ | 561 | 555 | 145 | 8 |
| 3 | 0.29 | $1.1 \cdot 10^{-16}$ | 143 | 89 | 66 | 8 |
| 4 | 2.32 | $6.8 \cdot 10^{-16}$ | 463 | 481 | 99 | 9 |
| 5 | 0.48 | $1.2 \cdot 10^{-15}$ | 233 | 108 | 148 | 9 |
| 6 | 7.96 | $1.9 \cdot 10^{-14}$ | 455 | 462 | 153 | 10 |
| 7 | 29.00 | $4.3 \cdot 10^{-15}$ | 1423 | 1543 | 247 | 13 |
| 8 | 1.01 | $1.1 \cdot 10^{-15}$ | 366 | 348 | 40 | 6 |
| 9 | 0.31 | $5.4 \cdot 10^{-16}$ | 157 | 81 | 86 | 8 |
| 10 | 1.25 | $1.1 \cdot 10^{-15}$ | 268 | 241 | 107 | 8 |

# A tricky example (life is not always so easy)

- Let $Z$ be the down-shift matrix having ones in the lower diagonal and zeros elsewhere, let $e_1 = (1, 0, 0, \ldots)^T$, $\mathbf{1}^T = (1, 1, \ldots)^T$
- Define $A_{-1} = e_1 e_1^T$, $A_0 = \frac{1}{2} Z$, $A_1 = \frac{1}{2}(I - e_1 e_1^T)$
- We have

$$R = \frac{1}{2} \left[ \begin{array}{c|c} 0 & 0 \\ \hline \mathbf{1} & (I - \frac{1}{2} Z)^{-1} \end{array} \right], \quad G = \mathbf{1} e_1^T$$

- $E_r$ and $E_g$ are neither in $\mathcal{F}$ nor have a bounded 2-norm ☹. They have bounded $\infty$-norm, but matrices in $L^\infty$ are not finitely representable, in general

**Remark:** The matrix $H_0$ is not invertible in $\mathcal{QT}$

# Exponential of a Toeplitz matrix

Let $A = T(a)$ and let

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

We have the following properties concerning the powers of $A$

$$\begin{aligned} A^k &= T(a^k) + E_k \\ E_k &= T(a)E_{k-1} + H(a_-)H((a^{k-1})_+), \quad k \geq 2 \\ E_1 &= 0 \end{aligned}$$

and

$$\|E_k\|_p \leq (k-1)\|a\|_{\mathcal{W}}^k, \qquad \|E_k\|_{\mathcal{F}} \leq \frac{k(k-1)}{2}\|a'\|_{\mathcal{W}}^2 \|a\|_{\mathcal{W}}^{k-2}$$

# Computing the exponential of a Toeplitz matrix

**Theorem**

*Let $a(z) \in \mathcal{W}_1$ and $A = T(a)$. Then the sequence $S_k = \sum_{i=0}^{k} \frac{1}{i!} A^i$ is a Cauchy sequence in $\mathcal{QT}$. There exists*

$$\exp(A) = \lim_k S_k = \sum_{i=0}^{\infty} \frac{1}{i!} A^i \in \mathcal{QT}$$

*Moreover,*

$$\exp(A) = T(\exp(a)) + E_{\exp}$$

*with $\|E_{\exp}\|_{\mathcal{F}} \leq \frac{1}{2} \|a'\|_{\mathcal{W}}^2 \exp(\|a\|_{\mathcal{W}})$*

An analogous result holds for the norm $\|A\|_{\mathcal{QT}_p} = \alpha \|a\|_{\mathcal{W}} + \|E\|_p$, or if $A = T(a) + E$.

# Extension to other functions

Recall that $T(a)^k = T(a^k) + E_k$

From the inequality $\|E_k\|_{\mathcal{F}} \leq \frac{k(k-1)}{2}\|a'\|_{\mathcal{W}}^2\|a\|_{\mathcal{W}}^{k-2}$ it follows

**Theorem**

*If $f(x) = \sum_{i=0}^{+\infty} f_i x^i$ is analytic for $|x| < \rho$ and if $a(z) \in \mathcal{W}_1$ is such that $\|a\|_{\mathcal{W}} < \rho$ then $f(a(z)) \in \mathcal{W}_1$ and*

$$f(T(a)) = T(f(a)) + F \in \mathcal{QT}$$

*with $\|F\|_{\mathcal{F}} \leq \frac{1}{2}\|a'\|_{\mathcal{W}}^2 f''(\|a\|_{\mathcal{W}})$*

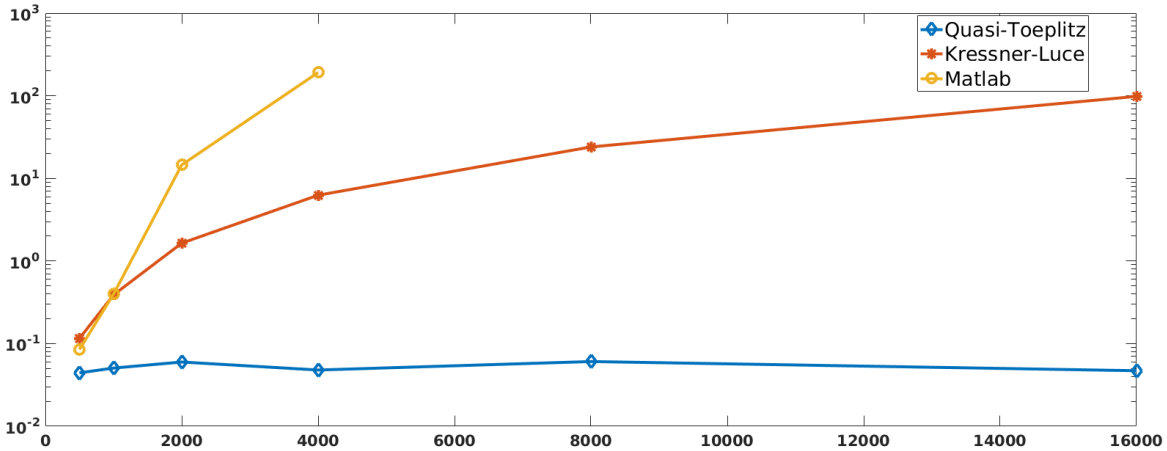
A similar result holds for $f(A)$, if $A = T(a) + E$

# Some numerical experiments (finite matrices)

Comparison with the toolbox by Kressner and Luce (SIMAX 2018) and Matlab

$a(z) = \sum_{i=-n}^{n} a_i z^i$, $a_{-i} = 0.7^i$, $a_i = (i+1)0.5^i$, $i \geq 0$,

CPU time:



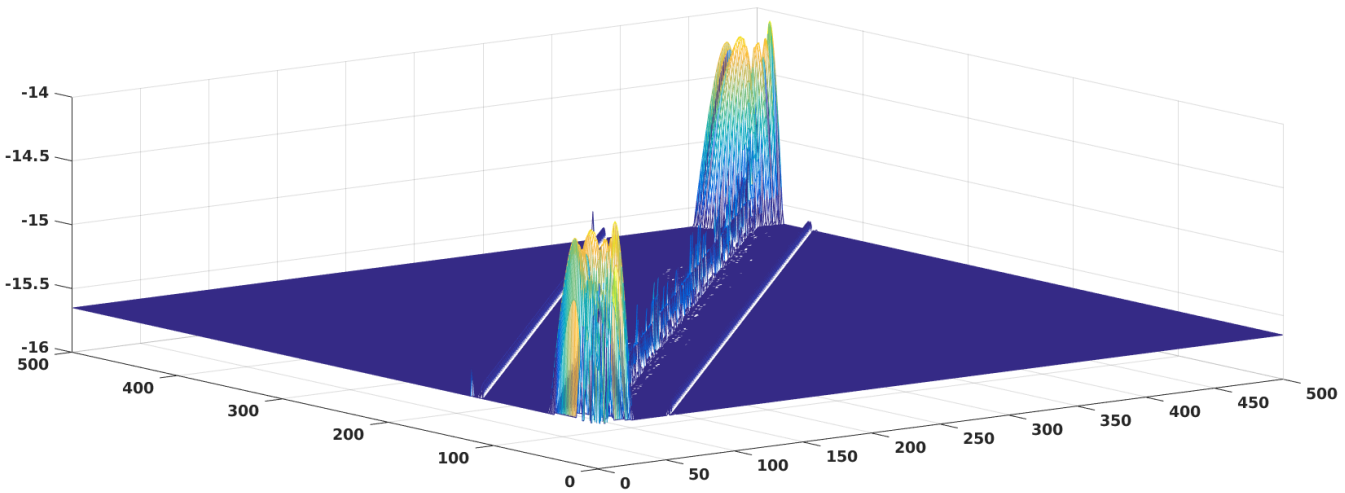

Merton problem (Kressner-Luce) CPU time

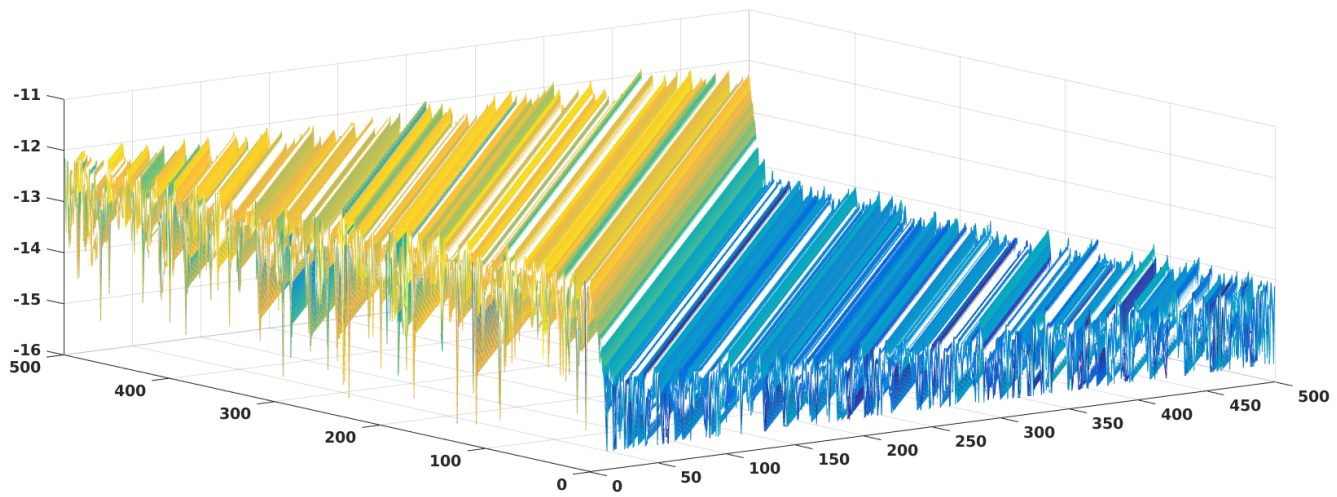
| n | 1000 | 2000 | 4000 | 8000 | 16000 | 32000 | 64000 |
|---|---|---|---|---|---|---|---|
| `expm` | 0.23 | 1.7 | 13.9 | 116 | - | - | - |
| KL | 0.3 | 0.8 | 3.4 | 12.9 | 48.2 | 185.9 | * |
| QT | 0.57 | 1.1 | 2.4 | 7.0 | 17.3 | 40.0 | 91.4 |

# Some numerical experiments (finite matrices)

Quasi-Toeplitz: error =`8.3e-15`;

Kressner-Luce: error =`1.3e-12`



# Research directions

- Consider matrices $A = T(a) + E$ where the correction $E = (e_{i,j}) \in L^\infty$ has columns not necessarily decaying to 0 but having a limit, i.e., matrices $E$ that can be written as $E = ev^T + \widehat{E}$, where $e = (1, 1, \ldots)^T$ and $\widehat{E}$ has columns decaying to 0. Thus $E$ is finitely representable, but:
  - is this a Banach algebra?
  - are the sequences generated by the available algorithms Cauchy?
  - how can we provide an effective finite representation of operators belonging to this space?
  - ...
- Provide a priori bounds to the numerical rank and size of the correction $E_g$ and $E_r$ in the solutions $G$ and $R$, and of the correction in $f(A)$.
- Provide a priori bounds to the decay properties of the Toeplitz parts in $G$, $R$ and $f(A)$.

# Some references

D.A. Bini, S. Massei, and B. Meini. On functions of quasi Toeplitz matrices. Sbornik Mathematics 2017

D.A. Bini, S. Massei, and B. Meini. Semi-infinite quasi-Toeplitz matrices with applications to QBD stochastic processes. Math. Comp. 2018

D.A. Bini, S. Massei, B. Meini, and L. Robol. On quadratic matrix equations with infinite size coefficients encountered in QBD stochastic processes. Num. Linear Alg. Appl. 2018.

A. Boettcher and S. M. Grusky. Spectral Properties of Band Toeplitz Matrices. SIAM, Philadelphia, PA, 2005.

A. Boettcher and B. Silbermann. Introduction to large truncated Toeplitz matrices. Springer 2012.

S. Dendievel and G. Latouche. Approximations for time-dependent distributions in Markovian fluid models. Meth. Comp. Appl. Prob. 2016

G. Fayolle, R. Iasnogorodski, and V. Malyshev. Random Walks in the Quarter-Plane. Springer, 1999.

D. Kressner and R. Luce. Fast computation of the matrix exponential for a Toeplitz matrix. SIMAX 2018.

A. J. Motyer and P. G. Taylor. Decay rates for Quasi-Birth-and-Death processes with countably many phases and tridiagonal block generators. Adv. Appl. Prob., 2006.

Thank you