Quasi-Toeplitz matrices: analysis, algorithms and applications

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Outline

Toeplitz matrices

Quasi-Toeplitz (QT) matrices

Quadratic matrix equations with QT coefficients

Functions of a QT matrix
Toeplitz matrices

A matrix $T = (t_{i,j})$ is Toeplitz if there exists a sequence $\{a_k\}_{k \in \mathbb{Z}}$ such that $t_{i,j} = a_{j-i}$

A Toeplitz matrix $T$ can be
- finite $n \times n$ if $i, j = 1, \ldots, n$
- semi-infinite if $i, j \in \mathbb{Z}^+$
- bi-infinite if $i, j \in \mathbb{Z}$

For instance,

$$
\begin{bmatrix}
  a_0 & a_1 \\
  a_1 & a_0 & a_1 \\
  a_0 & a_1 \\
  \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\begin{bmatrix}
  \ddots & \ddots & \ddots & \ddots \\
  \ddots & \ddots & \ddots & \ddots \\
  \ddots & \ddots & \ddots & \ddots \\
  \ddots & \ddots & \ddots & \ddots \\
  a_{-1} & a_0 & a_1 \\
\end{bmatrix},
\begin{bmatrix}
  \ddots & \ddots & \ddots & \ddots \\
  \ddots & \ddots & \ddots & \ddots \\
  \ddots & \ddots & \ddots & \ddots \\
  \ddots & \ddots & \ddots & \ddots \\
  a_{-1} & a_0 & a_1 \\
\end{bmatrix}
$$
An infinite Toeplitz matrix, or a sequence of $n \times n$ Toeplitz matrices, can be associated with a formal Laurent series $a(z) = \sum_{k=-\infty}^{+\infty} a_k z^k$.

In this case we write $T = T(a)$.

Observe that if $a(z)$ is a Laurent polynomial, say, $a(z) = \sum_{k=-1}^{2} a_k z^k$ then $T(a)$ is banded.

$$T(a) = \begin{bmatrix}
    a_0 & a_1 & a_2 \\
    a_{-1} & a_0 & a_1 & a_2 \\
    & a_{-1} & a_0 & a_1 & \ddots \\
    & & & \ddots & \ddots \\
    & & & & \ddots & \ddots
\end{bmatrix}$$
Toeplitz matrices in the applications

Toeplitz matrices are encountered in the mathematical models where some entity enjoys some shift invariance property.

For this reason, they are ubiquitous.

Typical examples

- FIR filters: the impulse response is shift invariant.
- Integral equations $\int_{a}^{b} k(x - t)f(t)dt = g(x)$
- Image restoration with shift invariant PSF.
- Queueing models where probability is shift-invariant.
- Finite differences discretization of derivatives, say $-f''(x)$ and trid$(-1,2,-1)$.
Problem 1: a bi-dimensional random walk

- In the quarter plane \( \{(i,j), \ i,j \in \mathbb{N}_+\} \), a particle can remain in the same position or move to the 8 neighbouring positions with given probabilities.

- These probabilities are independent of the position of the particle.

- When the particle reaches the boundaries \( i = 0 \) or \( j = 0 \), it is reflected with given probabilities.

**Applications:** Tandem queues with infinite buffer.
A bi-dimensional random walk (cont.)

Let $X_n$ be the position of the particle at time $n$.
Then $\{X_n\}_n$ is a Markov chain with transition probability matrix

$$
P = \begin{bmatrix}
B_0 & B_1 \\
A_{-1} & A_0 & A_1 \\
A_{-1} & A_0 & A_1 \\
\cdots & \cdots & \cdots
\end{bmatrix}
$$

where

$$
B_i = \begin{bmatrix}
\hat{b}_{0,i} & \hat{b}_{1,i} \\
b_{-1,i} & b_{0,i} & b_{1,i} \\
\cdots & \cdots & \cdots
\end{bmatrix}, \quad
A_i = \begin{bmatrix}
\hat{a}_{0,i} & \hat{a}_{1,i} \\
a_{-1,i} & a_{0,i} & a_{1,i} \\
\cdots & \cdots & \cdots
\end{bmatrix}
$$

and $B_0 + B_1$, $A_{-1} + A_0 + A_1$ row stochastic.

Structure: semi-infinite block tridiagonal, quasi block Toeplitz matrix with semi-infinite tridiagonal, quasi Toeplitz blocks
Problem: Compute, in the positive recurrent case, the invariant probability vector $\pi$, such that $\pi^T \mathcal{P} = \pi^T$ and $\|\pi\|_1 = 1$.

- Partition $\pi$ into blocks $\pi_i$, $\pi^T = [\pi_0^T, \pi_1^T, \pi_2^T, \ldots]$.
- The vector $\pi$ is such that $\pi_i^T = \pi_0^T R^i$, $i = 1, 2, \ldots$, where $R$ is the minimal nonnegative solution of the matrix equation

$$X = X^2 A_{-1} + X A_0 + A_1$$

- $\pi_0$ is such that $\pi_0^T (B_0 + B_1 G) = 0$ where $G$ is the minimal nonnegative solution of the matrix equation

$$X = A_{-1} + A_0 X + A_1 X^2$$
Questions

- How to solve these equations with infinite-dimensional block coefficients?
- Can we exploit the structure of the matrix coefficients?
- Do the solutions $G$ and $R$ have a (Toeplitz?) structure?

**Numerical evidence**: Log plot of a portion of the solution $G$
Numerical evidence

Log plot of the displacement shift
\[ \Delta = G(1 : n - 1, 1 : n - 1) - G(2 : n, 2 : n) \]

Numerically, the displacement shift of \( G \) goes exponentially to zero moving from the diagonal corners to the center.
Problem 2: Numerical solution of the heat equation

Solve
\[ \frac{\partial u(x, t)}{\partial t} = \gamma \frac{\partial^2 u(x, t)}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \quad \gamma > 0 \]
with boundary/initial conditions
\[ u(0, t) = u(1, t) = 0, \quad u(x, 0) = f(x). \]

The discretization of the second derivative by means of finite differences with the grid \( x_i = ih, \ i = 0, 1, \ldots, n + 1, \) with \( h = 1/(n + 1), \) leads to the IVP

\[ v'(t) = \gamma Av(t), \]
\[ v(0) = g \]

where \( v(t) = (v_i(t)), \ v_i(t) = u(x_i, t), \ g = (g_i), \ g_i = f(x_i), \)

\[ A = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \]
Matrix exponential

- A solution to this IVP is given by

\[ v(t) = \exp(\gamma t A)g \]

so that, setting \( t_j = j\Delta t \) for a given time step \( \Delta t \), we have

\[ v(t_j) = \exp(\gamma \Delta t A)v(t_{j-1}) \]

- Finite spatial domain, say \([0, 1]\), leads to a matrix exponential of a finite Toeplitz matrix

- Semi-infinite domain, say \([0, \infty)\), leads to a matrix exponential of a semi infinite Toeplitz matrix

**Applications:** mathematical models from finance, Black and Scholes equation, Merton integral differential equation \([\text{Merton 1976}], \ [\text{Kressner, Luce 2018}]\)
Questions

- If $T = T(a)$ is a Toeplitz matrix, is $\exp(T)$ Toeplitz as well? How “far” is $\exp(T)$ from $T(\exp(a))$?
- In general, given $f : \mathbb{C} \to \mathbb{C}$ and a Toeplitz matrix $T = T(a)$, is $f(T)$ Toeplitz as well? How “far” is $f(T)$ from $T(f(a))$?

**Numerical evidence**: Take $T = \text{trid}(1, 2, 1)$ of size $n = 100$, compute $B = \exp(T)$ with the function $\expm$ of Matlab and plot it
Numerical Evidence

Log plot of $|\Delta| = |B(1 : n - 1, 1 : n - 1) - B(2 : n, 2 : n)|$

Numerically the exponential is still Toeplitz except for the few entries in two opposite corners
Problem analysis

1. We introduce the class of Quasi-Toeplitz matrices, consisting of matrices of the form $T(a) + E$, where $T(a)$ is the Toeplitz part and $E$ is a “compact correction”

2. We introduce a matrix arithmetic and a Banach algebra containing these matrices

3. Matrices belonging to this algebra can be represented by means of a finite number of parameters, with a given representation error

4. We show that, under suitable assumptions, the sought solutions $G$ and $R$ of the quadratic equations, as well as the matrix function $f(T)$, with $T$ Toeplitz, are Quasi-Toeplitz matrices
Some definitions

- **Unit circle**: \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \)

- **Wiener class**: 
  \[ \mathcal{W} = \{ a(z) = \sum_{i \in \mathbb{Z}} a_i z^i : \mathbb{T} \to \mathbb{C}, \quad \sum_{i \in \mathbb{Z}} |a_i| < \infty \} \]
  \[ \|a\|_{\mathcal{W}} := \sum_{i \in \mathbb{Z}} |a_i| \]

- **Subset of \( \mathcal{W} \)**: 
  \[ \mathcal{W}_1 = \{ a(z) \in \mathcal{W} : a'(z) \in \mathcal{W} \} \text{ where } a'(z) = \sum_{i \in \mathbb{Z}} i a_i z^{i-1} \]
  \[ \|a\|_{\mathcal{W}_1} := \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}} \]

- **Semi-infinite Toeplitz matrix associated with** \( a(z) \in \mathcal{W} \): 
  \[ T(a) = (t_{i,j}), \quad t_{i,j} = a_{j-i}, \quad i, j \in \mathbb{Z}^+ \]
Some properties
Boettcher, Grudsky, Silbermann

- \( \mathcal{W} \) with the norm \( \| a \|_\mathcal{W} \) is a Banach algebra, that is,
  - a linear space
  - closed under multiplication
  - the norm is sub-multiplicative, i.e., \( \| ab \|_\mathcal{W} \leq \| a \|_\mathcal{W} \| b \|_\mathcal{W} \)
  - complete space, i.e., any Cauchy sequence converges in \( \mathcal{W} \)

- \( \mathcal{W}_1 \) with the norm \( \| a \|_{\mathcal{W}_1} \) is a Banach algebra

- \( \| T(a) \|_p \leq \| a \|_{\mathcal{W}}, \) for any \( p \geq 1 \), including \( p = \infty \)

\[
\| A \|_p = \sup_{\| x \|_p = 1} \| A x \|_p, \quad \| x \|_p = \left( \sum_{i \in \mathbb{Z}^+} |x_i|^p \right)^{1/p}
\]
**Theorem**

For $a(z), b(z) \in \mathcal{W}$, let $c(z) := a(z)b(z)$, then

$$T(a)T(b) = T(c) - H(a-)H(b_+),$$

where $H(a-) = (a_{-i-j+1})_{i,j \in \mathbb{Z}^+}$, $H(b_) = (b_{i+j-1})_{i,j \in \mathbb{Z}^+}$ are Hankel matrices

$$H(a-) = \begin{bmatrix}
a_{-1} & a_{-2} & a_{-3} & \ldots \\
a_{-2} & a_{-3} & \ddots & \vdots \\
a_{-3} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\end{bmatrix}, \quad H(b_+) = \begin{bmatrix}
b_1 & b_2 & b_3 & \ldots \\
b_2 & b_3 & \ddots & \vdots \\
b_3 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\end{bmatrix}$$

**In words**: The product of two infinite Toeplitz matrices differs from a Toeplitz matrix by a correction which is located mostly in the upper left corner.
A Banach algebra

The set of matrices

\[ \{ T(a) + K : a(z) \text{ is continuous in } \mathbb{T}, \ K \text{ is a compact operator in } \ell^2 \} \]

is a Banach algebra of operators in \( \ell^2 \) (Boettcher, Silbermann)
A Banach algebra

The set of matrices

\{ T(a)+K : a(z) \text{ is continuous in } \mathbb{T}, K \text{ is a compact operator in } \ell^2 \}

is a Banach algebra of operators in \( \ell^2 \) (Boettcher, Silbermann)

But:

- \( \| \cdot \|_2 \) cannot be easily computed
- we would like a norm such that the Toeplitz part and the correction are separated
- in our applications some matrices are not in \( \ell^2 \)
A Banach algebra

The set of matrices

\{ T(a) + K : a(z) \text{ is continuous in } \mathbb{T}, K \text{ is a compact operator in } \ell^2 \} 

is a Banach algebra of operators in \( \ell^2 \) (Boettcher, Silbermann)

But:

\begin{itemize}
  \item \( \| \cdot \|_2 \) cannot be easily computed
  \item we would like a norm such that the Toeplitz part and the correction are separated
  \item in our applications some matrices are not in \( \ell^2 \)
\end{itemize}

Therefore: we introduce a different Banach algebra
Quasi-Toeplitz (QT) matrices

- For $E = (e_{i,j})_{i,j \in \mathbb{Z}^+}$ define $\|E\|_\mathcal{F} := \sum_{i,j \in \mathbb{Z}^+} |e_{i,j}|$, and
  \[
  \mathcal{F} = \{ F = (f_{i,j})_{i,j \in \mathbb{Z}^+} : \|F\|_\mathcal{F} < +\infty \}
  \]

- Define **Quasi-Toeplitz (QT)** a matrix of the form
  \[
  A = T(a) + E, \quad a(z) \in \mathcal{W}_1, \quad E \in \mathcal{F}
  \]

- The set $\mathcal{QT}$ of QT matrices is a **Banach algebra** with the norm
  \[
  \|A\|_{\mathcal{QT}} = \|a\|_\mathcal{W} + \|a'\|_\mathcal{W} + \|E\|_\mathcal{F}
  \]
Quasi-Toeplitz (QT) matrices

- For $E = (e_{i,j})_{i,j \in \mathbb{Z}^+}$ define $\|E\|_\mathcal{F} := \sum_{i,j \in \mathbb{Z}^+} |e_{i,j}|$, and

$$\mathcal{F} = \{ F = (f_{i,j})_{i,j \in \mathbb{Z}^+} : \|F\|_\mathcal{F} < +\infty \}$$

- Define Quasi-Toeplitz (QT) a matrix of the form

$$A = T(a) + E, \quad a(z) \in \mathcal{W}_1, \quad E \in \mathcal{F}$$

- The set $\mathcal{QT}$ of QT matrices is a Banach algebra with the norm

$$\|A\|_{\mathcal{QT}} = \|a\|_{\mathcal{W}} + \|a'\|_{\mathcal{W}} + \|E\|_\mathcal{F}$$

Pros and cons:

😊 Computing $\|A\|_{\mathcal{QT}}$ is easy

😢 The norm is too restrictive, there are functions $a(z) \in \mathcal{W}$ such that $a'(z) \not\in \mathcal{W}$
A wider Banach algebra

For $A = T(a) + E$, with $a(z) \in \mathcal{W}$ and $E$ compact operator in $L^p$ define

$$
\|A\|_{QT_p} = \alpha \|a\|_{\mathcal{W}} + \|E\|_p, \quad \alpha = (1 + \sqrt{5})/2,
$$

Theorem

The space $QT_p$ formed by matrices $A$ such that $\|A\|_{QT_p} < \infty$ is a Banach algebra, moreover,

$$
QT \subset QT_p
$$
A wider Banach algebra

For $A = T(a) + E$, with $a(z) \in \mathcal{W}$ and $E$ compact operator in $L^p$ define

$$\|A\|_{QT_p} = \alpha \|a\|_{\mathcal{W}} + \|E\|_p, \quad \alpha = (1 + \sqrt{5})/2,$$

**Theorem**

The space $QT_p$ formed by matrices $A$ such that $\|A\|_{QT_p} < \infty$ is a Banach algebra, moreover,

$$QT \subset QT_p$$

**Pros and cons:**

😄 the class is wider than before

😊😊 if $p \notin \{1, \infty\}$, $E$ can be approximated by a matrix with a finite support. This is false if $p \in \{1, \infty\}$

😄 if $p = 2$, SVD can be used to manipulate and compress $E$
Pictorial representation of QT matrices

\[ A = \text{T}(a) + E \]

\[ A = \text{Toeplitz part} + \text{correction} \]
Some computational consequences

Any matrix $A = T(a) + E$ belonging to $QT$ or to $QT_p$, with $1 < p < \infty$, can be represented by means of a finite set of parameters to any desired precision:

- Since $a(z)$ is represented by a bi-infinite sequence $\{a_k\}_{k \in \mathbb{Z}}$ having decay of the coefficients, we may represent $a(z)$ with a finite sequence $a = (a_{n-}, \ldots, a_0, \ldots, a_{n+})$ up to an arbitrarily small error.

- The matrix $E$ is represented by means of the pair $(U, V)$ such that $E = UV^T$, $U$ and $V$ have a finite number of columns given by the numerical rank of $E$.

- The matrices $U$ and $V$ can be truncated to a finite number of rows.
Matrix arithmetic

- Relying on this representation, a **matrix arithmetic** can be defined and implemented for QT matrices.

\[
(T(a) + E_a) + (T(b) + E_b) = T(a + b) + E_{ab}
\]
\[
(T(a) + E_a)(T(b) + E_b) = T(ab) + F_{ab}
\]
\[
(T(a) + E_a)^{-1} = T(a^{-1}) + G_{ab}
\]

- The inversion is based on the **Wiener-Hopf factorization**
  \[ a(z) = u(z)\ell(z^{-1}) \]

- An SVD-based compression technique is introduced in order to **keep small the numerical rank** of the correction matrix
The QT matrix arithmetic has been implemented in a Matlab Toolbox by Leonardo Robol and Stefano Massei
https://github.com/numpi/cqt-toolbox

The operation are invoked with the standard Matlab syntax, say,

\[ C = A + B, \quad C = A * B, \quad C = A \backslash B, \quad C = A / B \]

Other functions are available like

\[ [U, L] = ul(A), \quad B = inv(A), \quad A = qt(am, ap, E) \]
We define **Quasi Toeplitz (QT)** any matrix belonging to $QT$ or to $QT_p$

**Question:** If the coefficients of a quadratic matrix equation $A_{-1} + A_0X + A_1X^2 = X$ are QT matrices, is the solution a QT matrix?
A general result

Let $\mathcal{B}$ be a Banach algebra and $A_i \in \mathcal{B}$ for $i = -1, 0, 1$.

**Theorem**

Assume that $\varphi(z) = z^{-1}A_{-1} + (A_0 - I) + zA_1$ is invertible for $t^{-1} < |z| < t$, with $t > 1$, and let $\psi(z) = \varphi(z)^{-1} = \sum_{i \in \mathbb{Z}} z^i H_i$. If $H_0$ is invertible then $G = H_{-1}H_0^{-1}$ and $R = H_0^{-1}H_1$ solve

$$A_{-1} + A_0X + A_1X^2 = X \quad \text{and} \quad X^2A_{-1} + XA_0 + A_1 = X$$

respectively, they belong to $\mathcal{B}$ and their spectrum lies inside the open unit disk. Moreover

$$\varphi(z) = (I - zR)W(I - z^{-1}G)$$

with $W \in \mathcal{B}$ invertible.

**Remark:** In our framework, $\mathcal{B}$ is the Banach algebra of QT matrices, hence $G = T(g) + E_g$ and $R = T(r) + E_r$.
Solving the quadratic matrix equations

- **Idea:** Use the QT-arithmetic to extend standard algorithms, used in the finite dimensional case, to the case of QT matrix coefficients.

- We provide algorithms for computing $G$ and $R$, more precisely, for computing the coefficients of $g(z)$, $r(z)$ together with “slim” matrices $U$, $V$, $Y$, $W$ such that

  $$E_g = UV^T, \quad E_r = YW^T$$
Fixed point iteration

- Represent \( A_i = T(a_i) + E_i \), \( i = -1, 0, 1 \)
- Set \( G_0 = T(1) \) (e.g., \( G_0 \) is the identity matrix)
- For \( k = 1, 2, \ldots \) compute \( G_k = T(g_k) + E_k^{(g)} \) by means of

\[
G_k = A_{-1} + A_0 G_{k-1} + A_1 G_{k-1}^2
\]

until convergence

**Remark:** Linear convergence, sometimes very slow
Cyclic reduction

- Compute

\[
\begin{align*}
A_0^{(h+1)} &= A_0^{(h)} + A_1^{(h)} S^{(h)} A_{-1} + A_{-1}^{(h)} S^{(h)} A_1^{(h)}, \\
A_1^{(h+1)} &= A_1^{(h)} S^{(h)} A_1^{(h)}, \\
\hat{A}^{(h+1)} &= \hat{A}^{(h)} + A_{-1}^{(h)} S^{(h)} A_1^{(h)}, \\
\tilde{A}^{(h+1)} &= \tilde{A}^{(h)} + A_1^{(h)} S^{(h)} A_{-1},
\end{align*}
\]

with \( A_0^{(0)} = \tilde{A}^{(0)} = \hat{A}^{(0)} = A_0, A_1^{(0)} = A_1, A_{-1}^{(0)} = A_{-1} \), where

\[
A_i^{(h)} = T(a_i^{(h)}) + E_i^{(h)}, \text{ for } i = -1, 0, 1.
\]

- Set \( G_h = (I - \hat{A}^{(h)})^{-1} A_{-1}, R_h = A_1 (I - \tilde{A}^{(h)})^{-1} \)

**Remark:** Relying on the QT matrix arithmetic, cyclic reduction can be easily implemented. Under mild conditions the sequences \( \{G_h\} \) and \( \{R_h\} \) converge quadratically to \( G \) and \( R \), respectively
Some numerical tests with CR

Ten instances of the two-node Jackson network from [Motyer and Taylor 2006]: problems 1–10

\[
A_{-1} = \begin{bmatrix}
(1 - q)\mu_2 & q\mu_2 \\
(1 - q)\mu_2 & q\mu_2 \\
& \\
& \ddots & \ddots
\end{bmatrix}
\]

\[
A_0 = \begin{bmatrix}
-(\lambda_1 + \lambda_2 + \mu_2) & \lambda_1 \\
(1 - p)\mu_1 & -(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) & \lambda_1 \\
& \ddots & \ddots & \ddots
\end{bmatrix}
\]

\[
A_1 = \begin{bmatrix}
\lambda_2 \\
\rho\mu_1 & \lambda_2 \\
& \ddots & \ddots
\end{bmatrix}
\]
## Some numerical tests

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A tricky example (life is not always so easy)

Let $Z$ be the down-shift matrix having ones in the lower diagonal and zeros elsewhere, let $e_1 = (1, 0, 0, \ldots)^T$, $1^T = (1, 1, \ldots)^T$

Define $A_{-1} = e_1 e_1^T$, $A_0 = \frac{1}{2} Z$, $A_1 = \frac{1}{2}(I - e_1 e_1^T)$

We have

$$R = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 1 \end{bmatrix} (I - \frac{1}{2} Z)^{-1}, \quad G = 1 e_1^T$$

$E_r$ and $E_g$ are neither in $F$ nor have a bounded 2-norm 😞. They have bounded $\infty$-norm, but matrices in $L^\infty$ are not finitely representable, in general

Remark: The matrix $H_0$ is not invertible in $QT$
Exponential of a Toeplitz matrix

Let $A = T(a)$ and let

$$\exp(A) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i$$

We have the following properties concerning the powers of $A$

$$A^k = T(a^k) + E_k$$
$$E_k = T(a)E_{k-1} + H(a_-)H((a^{k-1})_+), \quad k \geq 2$$
$$E_1 = 0$$

and

$$\|E_k\|_p \leq (k - 1)\|a\|_W^k, \quad \|E_k\|_F \leq \frac{k(k - 1)}{2} \|a'\|_W^2 \|a\|_W^{k-2}$$
Computing the exponential of a Toeplitz matrix

**Theorem**

Let \( a(z) \in \mathcal{W}_1 \) and \( A = T(a) \). Then the sequence \( S_k = \sum_{i=0}^{k} \frac{1}{i!} A^i \) is a Cauchy sequence in \( QT \). There exists

\[
\exp(A) = \lim_{k} S_k = \sum_{i=0}^{\infty} \frac{1}{i!} A^i \in QT
\]

Moreover,

\[
\exp(A) = T(\exp(a)) + E_{\exp}
\]

with \( \| E_{\exp} \|_{\mathcal{F}} \leq \frac{1}{2} \| a' \|_{\mathcal{W}}^2 \exp(\| a \|_{\mathcal{W}}) \)

An analogous result holds for the norm \( \| A \|_{QT_p} = \alpha \| a \|_{\mathcal{W}} + \| E \|_{p} \), or if \( A = T(a) + E \).
Extension to other functions

Recall that $T(a)^k = T(a^k) + E_k$

From the inequality $\|E_k\|_F \leq \frac{k(k-1)}{2} \|a'\|_W^2 \|a\|_W^{k-2}$ it follows

**Theorem**

If $f(x) = \sum_{i=0}^{+\infty} f_i x^i$ is analytic for $|x| < \rho$ and if $a(z) \in \mathcal{W}_1$ is such that $\|a\|_W < \rho$ then $f(a(z)) \in \mathcal{W}_1$ and

$$f(T(a)) = T(f(a)) + F \in QT$$

with $\|F\|_F \leq \frac{1}{2} \|a'\|_W^2 f''(\|a\|_W)$

A similar result holds for $f(A)$, if $A = T(a) + E$
Some numerical experiments (finite matrices)

Comparison with the toolbox by Kressner and Luce (SIMAX 2018) and Matlab

\[ a(z) = \sum_{i=-n}^{n} a_i z^i, \quad a_{-i} = 0.7^i, \quad a_i = (i + 1)0.5^i, \quad i \geq 0, \]

CPU time:

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</table>
Some numerical experiments (finite matrices)

Quasi-Toeplitz: error = $8.3 \times 10^{-15}$;
Kressner-Luce: error = $1.3 \times 10^{-12}$
Research directions

- Consider matrices $A = T(a) + E$ where the correction $E = (e_{i,j}) \in L^\infty$ has columns not necessarily decaying to 0 but having a limit, i.e., matrices $E$ that can be written as $E = ev^T + \hat{E}$, where $e = (1, 1, \ldots)^T$ and $\hat{E}$ has columns decaying to 0. Thus $E$ is finitely representable, but:
  - is this a Banach algebra?
  - are the sequences generated by the available algorithms Cauchy?
  - how can we provide an effective finite representation of operators belonging to this space?
  - ...

- Provide **a priori bounds** to the numerical rank and size of the correction $E_g$ and $E_r$ in the solutions $G$ and $R$, and of the correction in $f(A)$.
- Provide **a priori bounds** to the decay properties of the Toeplitz parts in $G$, $R$ and $f(A)$. 

Some references


Thank you