Subspace methods for three-parameter eigenvalue problems

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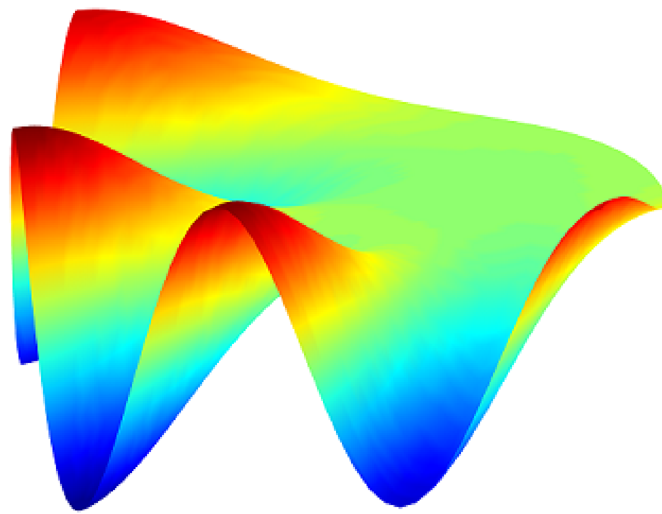

joint work with

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Karl Meerbergen (KU Leuven),
Emre Mengi (Koç University)

Introductory example

We consider the Helmholtz equation $\nabla^2 u + \omega^2 u = 0$, $u|_{\delta\Omega} = 0$, on the ellipse

$$\Omega = \{(x/\alpha)^2 + (y/\beta)^2 \le 1\}.$$

If we use elliptic coordinates and separation of variables we get a system of modified Mathieu's and Mathieu's DE

$$\boxed{\begin{array}{ll} F''(\xi) - (\lambda - 2\mu\cosh(2\xi))F(\xi) = 0, & F(0) = F(\xi_0) = 0, \\ G''(\eta) + (\lambda - 2\mu\cos(2\eta))G(\eta) = 0, & G(0) = G(\pi/2) = 0, \end{array}}$$

where $h = \sqrt{\alpha^2 - \beta^2}$, $\xi_0 = \text{arccosh}\dfrac{\alpha}{h}$, and $\mu = h^2\omega^2/4$.

This is a two-parameter eigenvalue problem (2EP).

Multiparameter eigenvalue problem (MEP)

In several coordinate systems, when separation of variables is applied to a PDE (Helmholtz, Laplace, Schrödinger,...), we obtain a MEP. A general form is

$$p_j(t_j)\, y_j''(t_j) + q_j(t_j)\, y_j'(t_j) + r_j(t_j)\, y_j(t_j) = \sum_{\ell=1}^{k} \lambda_\ell\, s_{j\ell}(t_j)\, y_j(t_j), \quad j = 1, \ldots, k,$$

where $t_j \in [a_j, b_j]$, with the appropriate b.c. We are looking for $(\lambda_1, \ldots, \lambda_k)$ and nontrivial functions $y_1, \ldots, y_k$ that satisfy the above equations and b.c.

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Discretization (e.g., Chebyshev collocation) leads to an algebraic MEP

$$\begin{array}{rcl} A_{10}\, x_1 & = & \lambda_1\, A_{11}\, x_1 + \cdots + \lambda_k\, A_{1k}\, x_1, \\ & \vdots & \\ A_{k0}\, x_k & = & \lambda_1\, A_{k1}\, x_k + \cdots + \lambda_k\, A_{kk}\, x_k, \end{array} \qquad \text{(MEP)}$$

where $A_{ij} \in \mathbb{C}^{n \times n}$ for $i = 1, \ldots, k$ and $j = 0, \ldots, k$.

- eigenvalue: $(\lambda_1, \ldots, \lambda_k)$, that satisfies (MEP) for nonzero $x_1, \ldots, x_k$,
- eigenvector: $x_1 \otimes \cdots \otimes x_k$.

Generically, the above (MEP) has n^k eigenvalues.

Operator determinants

$$\begin{aligned} A_{10}\, x_1 &= \lambda_1\, A_{11}\, x_1 + \cdots + \lambda_k\, A_{1k}\, x_1, \\ &\vdots \\ A_{k0}\, x_k &= \lambda_1\, A_{k1}\, x_k + \cdots + \lambda_k\, A_{kk}\, x_k, \end{aligned} \qquad \text{(MEP)}$$

We can solve (MEP) by introducing operator determinants of size $n^k \times n^k$

$$\Delta_0 = \begin{vmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{vmatrix}_{\otimes}$$

and

$$\Delta_i = \begin{vmatrix} A_{11} & \cdots & A_{1,i-1} & A_{10} & A_{1,i+1} & \cdots & A_{1k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_{k1} & \cdots & A_{k,i-1} & A_{k0} & A_{k,i+1} & \cdots & A_{kk} \end{vmatrix}_{\otimes}$$

for $i = 1, \ldots, k$. If Δ_0 is nonsingular, then matrices $\Delta_0^{-1}\Delta_1, \ldots, \Delta_0^{-1}\Delta_k$ commute and (MEP) is equivalent to a system of GEP

$$\begin{aligned} \Delta_1\, z &= \lambda_1\, \Delta_0\, z, \\ &\vdots \\ \Delta_k\, z &= \lambda_k\, \Delta_0\, z, \end{aligned}$$

for $z = x_1 \otimes \cdots \otimes x_k$ (Atkinson '72).

Numerical computation of eigenmodes

If MEP originates from a k-dimensional Helmholtz equation $\nabla^2 u + \omega^2 u = 0$, usually only one of the parameters $\lambda_1, \ldots, \lambda_k$ is related to the eigenfrequency ω.

Suppose that we want the eigenvalues (λ, μ) of 2EP

$$\begin{aligned} A_1 x_1 &= \lambda B_1 x_1 + \mu C_1 x_1 \\ A_2 x_2 &= \lambda B_2 x_2 + \mu C_2 x_2 \end{aligned}$$

with the smallest value of $|\mu|$.

Size: S

When n is small, we solve the related GEP $\Delta_2 z = \mu \Delta_0 z$, where

$$\begin{aligned} \Delta_0 &= B_1 \otimes C_2 - C_1 \otimes B_2 \\ \Delta_2 &= B_1 \otimes A_2 - A_1 \otimes B_2. \end{aligned}$$

We can do similar for 3EP when n is small.

Size: M (2EP) Arnoldi and Sylvester

For small eigenvalues of $\Delta_2 z = \mu \Delta_0 z$ we can apply IRA or Krylov–Schur to $\Delta_2^{-1}\Delta_0$. When building the Krylov subspace

$$\mathcal{K}_k(\Delta_2^{-1}\Delta_0, z_0) := \mathrm{Lin}(z_0, \Delta_2^{-1}\Delta_0\, z_0, \ldots, (\Delta_2^{-1}\Delta_0)^{k-1} z_0)$$

the most expensive step is the linear system $\Delta_2 w = \Delta_0 z$, equal to

$$\boxed{(B_1 \otimes A_2 - A_1 \otimes B_2)w = (B_1 \otimes C_2 - C_1 \otimes B_2)z.}$$

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Using the well-known equality

$$\boxed{(A \otimes B)\mathrm{vec}(X) = \mathrm{vec}(BXA^T)}$$

we write the system as a Sylvester equation

$$\boxed{B_2 W A_1^T - A_2 W B_1^T = C_2 Z B_1^T - B_2 Z C_1^T,}$$

where $w = \mathrm{vec}(W)$, $z = \mathrm{vec}(Z)$, and solve it in $\mathcal{O}(n^3)$ by, e.g., the Bartels–Stewart method (Meerbergen, P. '15).

Size: M Extension to 3EP

We want the eigenvalues (λ, μ, η) of

$$\begin{aligned} A_1 x_1 &= \lambda B_1 x_1 + \mu C_1 x_1 + \eta D_1 x_1 \\ A_2 x_2 &= \lambda B_2 x_2 + \mu C_2 x_2 + \eta D_2 x_2 \\ A_3 x_3 &= \lambda B_3 x_3 + \mu C_3 x_3 + \eta D_3 x_3 \end{aligned}$$

with the smallest value of $|\eta|$.

Similarly to 2EP we would like to apply IRA to $\Delta_3^{-1}\Delta_0$, where

$$\Delta_0 = \begin{vmatrix} B_1 & C_1 & D_1 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{vmatrix}_{\otimes}, \quad \Delta_3 = \begin{vmatrix} B_1 & C_1 & A_1 \\ B_2 & C_2 & A_2 \\ B_3 & C_3 & A_3 \end{vmatrix}_{\otimes}.$$

Open problem: Can we solve $\Delta_3 w = \Delta_0 z$ in less than $\mathcal{O}(n^9)$ flops?

We are not aware of a such method, even when all matrices B_i, C_i, D_i are diagonal.

We can apply IRA or Krylov-Schur, but only for a small n.

Size: L (2EP)

We cannot use subspace methods with full vectors of size n^2 for $\Delta_2 z = \mu \Delta_0 z$.

Instead, we use low-rank approximations and search subspaces on

$$\begin{aligned} A_1 x_1 &= \lambda B_1 x_1 + \mu C_1 x_1, \\ A_2 x_2 &= \lambda B_2 x_2 + \mu C_2 x_2. \end{aligned}$$

If columns of $U_k \otimes V_k$ span the search space, then Ritz pairs $((\sigma, \tau), U_k c \otimes V_k d)$ are solutions of the projected 2EP

$$\begin{aligned} U_k^H A_1 U_k c &= \sigma U_k^H B_1 U_k c + \tau U_k^H C_1 U_k c \\ V_k^H A_2 V_k d &= \sigma V_k^H B_2 V_k d + \tau V_k^H C_2 V_k d. \end{aligned}$$

- Jacobi-Davidson (Hochstenbach, Košir, P. '05), (Hochstenbach, P. '08)
  - good when the target is a point (λ_0, μ_0), works also for a target line $\mu = \mu_0$.
- Subspace iteration with Arnoldi expansion (Meerbergen, P. '15)
  - suitable when the target is a line $\mu = \mu_0$,

Extension to 3EP: (Hochstenbach, Meerbergen, Mengi, P. '18)

Algorithm 1: Jacobi–Davidson

1. Initial matrices $U_0^{(j)} \in \mathbb{C}^{n\times\ell}$ with orthonormal columns for $j = 1, 2, 3$

   for $k = 0, 1, \ldots$

2. Extract appropriate Ritz pair $((\sigma, \tau, \psi), U_k^{(1)}s_1 \otimes U_k^{(2)}s_2 \otimes U_k^{(3)}s_3)$ of

$$\begin{aligned}
{U_k^{(1)}}^H A_1 U_k^{(1)} s_1 &= \sigma {U_k^{(1)}}^H B_1 U_k^{(1)} s_1 + \tau {U_k^{(1)}}^H C_1 U_k^{(1)} s_1 + \psi {U_k^{(1)}}^H D_1 U_k^{(1)} s_1,\\
{U_k^{(2)}}^H A_2 U_k^{(2)} s_2 &= \sigma {U_k^{(2)}}^H B_2 U_k^{(2)} s_2 + \tau {U_k^{(2)}}^H C_2 U_k^{(2)} s_2 + \psi {U_k^{(2)}}^H D_2 U_k^{(2)} s_2,\\
{U_k^{(2)}}^H A_3 U_k^{(3)} s_3 &= \sigma {U_k^{(3)}}^H B_3 U_k^{(3)} s_3 + \tau {U_k^{(3)}}^H C_3 U_k^{(3)} s_3 + \psi {U_k^{(3)}}^H D_3 U_k^{(3)} s_3.
\end{aligned}$$

3. Refine Ritz pair
4. $r_j = (A_j - \sigma B_j - \tau C_j - \psi D_j)u_j$, where $u_j = U_k^{(j)} s_j$, for $j = 1, 2, 3$
5. Extract eigenpair if $(\|r_1\|^2 + \|r_2\|^2 + \|r_3\|^2)^{1/2} \le \varepsilon$
6. Solve for $j = 1, 2, 3$ (approx. or exactly) the correction equation

   $(I - u_j u_j^H)(A_j - \sigma B_j - \tau C_j - \psi D_j)v_j = -r_j, \qquad v_j \perp u_j$

7. Expand $U_k^{(j)} = \mathrm{RGS}(U_{k-1}^{(j)}, s_j)$ for $j = 1, 2, 3$

Missing details: targeting, selection criteria, refinement, restarts, ...

Block Arnoldi method for the Sylvester equation (2EP)

(Hu, Reichel '92) A method for a Sylvester equation $AX - XB = C$ such that $C = GF^T$, where F and G have just few columns.

- Build orthogonal basis U_k for $\mathcal{K}_k(A, F)$.
- Build orthogonal basis V_k for $\mathcal{K}_k(B^T, G)$.
- An approximate low-rank solution is $X_k = U_k D_k V_k^H$, where
$$(U_k^H A U_k) D_k - D_k (V_k^H B V_k) = U_k^H C V_k.$$

Recall that $\Delta_2 w = \Delta_0 z$ is equivalent to the Sylvester equation
$$\widetilde{B}_2 W - W \widetilde{B}_1^T = \widetilde{C}_2 Z \widetilde{B}_1^T - \widetilde{B}_2 Z \widetilde{C}_1^T.$$
where $w = \mathrm{vec}(W)$, $z = \mathrm{vec}(Z)$, $\widetilde{B}_i = A_i^{-1} B_i$, and $\widetilde{C}_i = A_i^{-1} C_i$.

If $z \in \mathrm{span}(U \otimes V)$ then the righthand side lies in $\mathrm{span}(F \otimes G)$, where
$$F = [\, A_1^{-1} B_1 U \quad A_1^{-1} C_1 U \,] \quad \text{and} \quad G = [\, A_2^{-1} B_2 V \quad A_2^{-1} C_2 V \,],$$
and a low-rank approximation for w lies in
$$\mathcal{K}_r(A_1^{-1} B_1, F) \otimes \mathcal{K}_r(A_2^{-1} B_2, G).$$

Extension to 3EP

The linear system for the expansion is $\Delta_3 w = \Delta_0 z$, where

$$\Delta_0 = \left| \begin{matrix} B_1 & C_1 & D_1 \\ B_2 & C_2 & D_2 \\ B_3 & C_3 & D_3 \end{matrix} \right|_{\otimes}, \quad \Delta_3 = \left| \begin{matrix} B_1 & C_1 & A_1 \\ B_2 & C_2 & A_2 \\ B_3 & C_3 & A_3 \end{matrix} \right|_{\otimes}.$$

For $z \in \operatorname{span}(U_1 \otimes U_2 \otimes U_3)$ the righthand side lies in $\operatorname{span}(F_1 \otimes F_2 \otimes F_3)$, where

$$F_i = [\, A_i^{-1} B_i U_i \quad A_i^{-1} C_i U_i \quad A_i^{-1} D_i U_i \,],$$

and we search a low-rank approximation for w in

$$\mathcal{K}_r(A_1^{-1}B_1, A_1^{-1}C_1, F_1) \otimes \mathcal{K}_r(A_2^{-1}B_2, A_2^{-1}C_2, F_2) \otimes \mathcal{K}_r(A_3^{-1}B_3, A_3^{-1}C_3, F_3),$$

where

$$\mathcal{K}_r(B, C, F) := \operatorname{span}(F, BF, CF, B^2F, BCF, CBF, C^2F, \ldots)$$

is a generalized Krylov subspace for $i = 1, 2, 3$.

- Subspace grows much faster than for 2EP, cannot use many Arnoldi steps.
- In the Arnoldi extension we keep only important new directions.

Algorithm 2: Subspace iteration with Arnoldi expansion

1. Initial matrices $U_0^{(j)} \in \mathbb{C}^{n\times\ell}$ with orthonormal columns for $j = 1,2,3$

   for $k = 0, 1, \ldots$

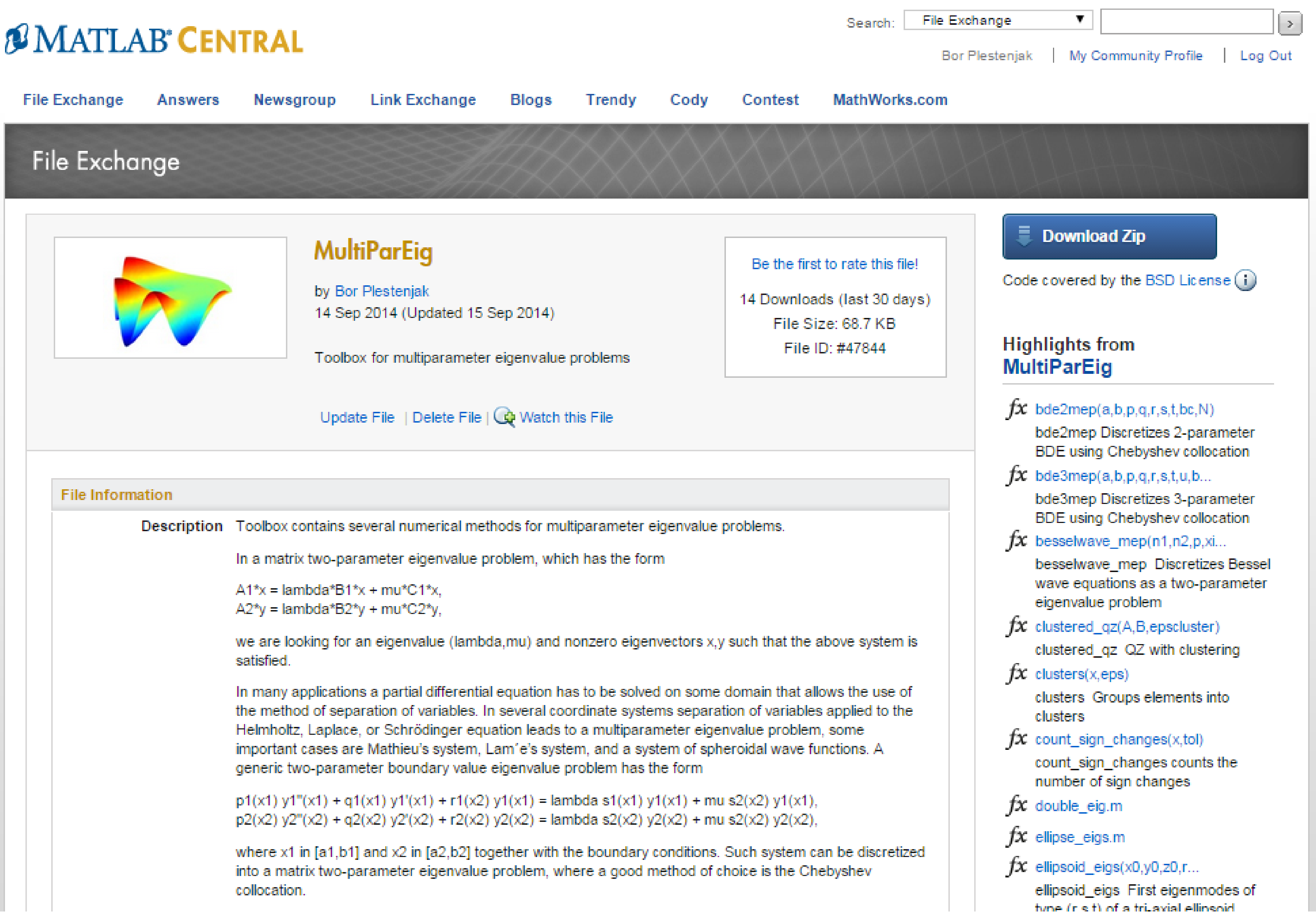
2. $F_j = \left[A_j^{-1}B_jU_k^{(j)} \;\; A_j^{-1}C_jU_k^{(j)} \;\; A_j^{-1}D_jU_k^{(j)}\right]$ for $j = 1,2,3$
3. Form Q_j with orthonormal basis for $\mathcal{K}_r(A_j^{-1}B_j, A_j^{-1}C_j, F_j)$ for $j = 1,2,3$
4. Extract $m > \ell$ appr. Ritz pairs $((\sigma_i, \tau_i, \psi_i), Q_1s_i^{(1)} \otimes Q_2s_i^{(2)} \otimes Q_3s_i^{(3)})$ of

$$\begin{aligned}
{Q_1}^H A_1 Q_1 s^{(1)} &= \sigma {Q_1}^H B_1 Q_1 s^{(1)} + \tau {Q_1}^H C_1 Q_1 s^{(1)} + \psi {Q_1}^H D_1 Q_1 s^{(1)},\\
{Q_2}^H A_2 Q_2 s^{(2)} &= \sigma {Q_2}^H B_2 Q_2 s^{(2)} + \tau {Q_2}^H C_2 Q_2 s^{(2)} + \psi {Q_2}^H D_2 Q_2 s^{(2)},\\
{Q_3}^H A_3 Q_3 s^{(3)} &= \sigma {Q_3}^H B_3 Q_3 s^{(3)} + \tau {Q_3}^H C_3 Q_3 s^{(3)} + \psi {Q_3}^H D_3 Q_3 s^{(3)}.
\end{aligned}$$

5. Refine Ritz pairs
6. Extract new Ritz pairs with small residuals
7. Form $U_{k+1}^{(j)}$ for $j = 1,2,3$ with orthonormal columns from the first ℓ remaining Ritz pairs that meet the selection criterion

Missing details: Arnoldi with SVD filtering, subspace shrinking, selection criteria, refinement

Numerical results - toolbox MultiParEig



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MultiParEig

by Bor Plestenjak

14 Sep 2014 (Updated 15 Sep 2014)

Toolbox for multiparameter eigenvalue problems

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File Information

Description Toolbox contains several numerical methods for multiparameter eigenvalue problems.

In a matrix two-parameter eigenvalue problem, which has the form

A1*x = lambda*B1*x + mu*C1*x,
A2*y = lambda*B2*y + mu*C2*y,

we are looking for an eigenvalue (lambda,mu) and nonzero eigenvectors x,y such that the above system is satisfied.

In many applications a partial differential equation has to be solved on some domain that allows the use of the method of separation of variables. In several coordinate systems separation of variables applied to the Helmholtz, Laplace, or Schrödinger equation leads to a multiparameter eigenvalue problem, some important cases are Mathieu's system, Lam´e's system, and a system of spheroidal wave functions. A generic two-parameter boundary value eigenvalue problem has the form

p1(x1) y1''(x1) + q1(x1) y1'(x1) + r1(x2) y1(x1) = lambda s1(x1) y1(x1) + mu s2(x2) y1(x1),
p2(x2) y2''(x2) + q2(x2) y2'(x2) + r2(x2) y2(x2) = lambda s2(x2) y2(x2) + mu s2(x2) y2(x2),

where x1 in [a1,b1] and x2 in [a2,b2] together with the boundary conditions. Such system can be discretized into a matrix two-parameter eigenvalue problem, where a good method of choice is the Chebyshev collocation.

Highlights from MultiParEig

fx bde2mep(a,b,p,q,r,s,t,bc,N)
bde2mep Discretizes 2-parameter BDE using Chebyshev collocation

fx bde3mep(a,b,p,q,r,s,t,u,b...
bde3mep Discretizes 3-parameter BDE using Chebyshev collocation

fx besselwave_mep(n1,n2,p,xi...
besselwave_mep Discretizes Bessel wave equations as a two-parameter eigenvalue problem

fx clustered_qz(A,B,epscluster)
clustered_qz QZ with clustering

fx clusters(x,eps)
clusters Groups elements into clusters

fx count_sign_changes(x,tol)
count_sign_changes counts the number of sign changes

fx double_eig.m

fx ellipse_eigs.m

fx ellipsoid_eigs(x0,y0,z0,r...
ellipsoid_eigs First eigenmodes of type (r,s,t) of a tri-axial ellipsoid

MultiParEig in action

We want the eigenmodes with the smallest $|\mu|$ of

$$\boxed{\begin{aligned} \xi^2 M''(\xi) + \xi M'(\xi) - M(\xi) &= -\lambda\xi^2 M(\xi) - \mu\xi^4 M(\xi), \\ \eta^2 N''(\eta) + \eta N'(\eta) - N(\eta) &= \quad \lambda\eta^2 N(\eta) - \mu\eta^4 N(\eta), \end{aligned}}$$

where $0 < \xi < \xi_0 = 1$, $0 < \eta < \eta_0 = 1$, and $M'(0) = M(1) = N'(0) = N(1) = 0$.

(2EP is related to the Helmholtz equation in parabolic rotational coordinates)

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where $0 < \xi < \xi_0 = 1$, $0 < \eta < \eta_0 = 1$, and $M'(0) = M(1) = N'(0) = N(1) = 0$.

(2EP is related to the Helmholtz equation in parabolic rotational coordinates)

```
p1 = @(x)  x.^2;      q1 = @(x) x;       r1 = -1;
s1 = @(x) -x.^2;      t1 = @(x) -x.^4;
[z1,A1,B1,C1,G1,k1,r1] = bde2mep(0,1,p1,q1,r1,s1,t1,[0 1;1 0],60);

p2 = @(y)  y.^2;      q2 = @(y)  y;      r2 = -1;
s2 = @(y)  y.^2;      t2 = @(y) -y.^4;
[z2,A2,B2,C2,G2,k2,r2] = bde2mep(0,1,p2,q2,r2,s2,t2,[0 1;1 0],60);

[lambda,mu] = twopareigs(A1,B1,C1,A2,B2,C2,6); sqrt(mu)'

ans =
    6.2832    9.3565    9.3565   12.3620   12.3620   12.5664
```

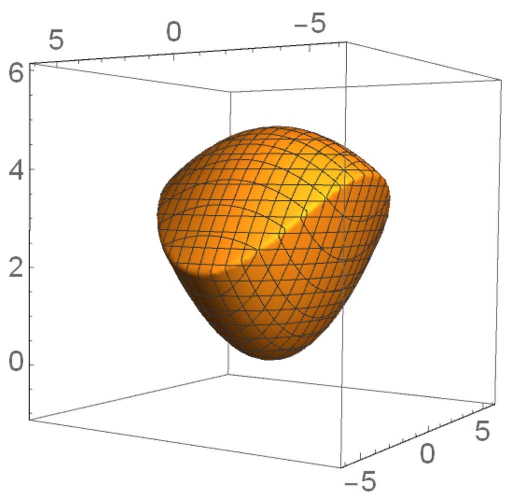
Example 1: Helmholtz equation in paraboloidal coordinates

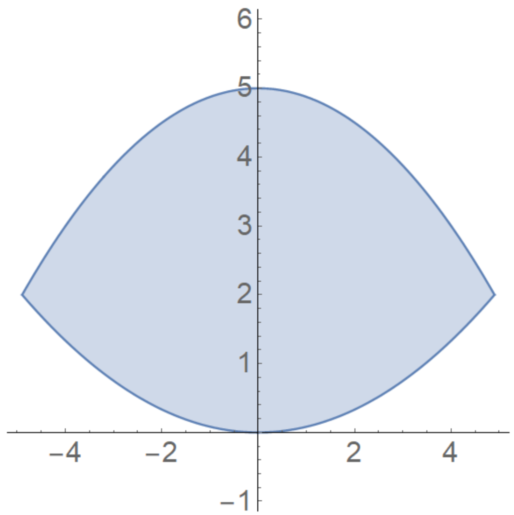
The Cartesian coordinates are related to paraboloidal coordinates (ξ_1, ξ_2, ξ_3) as

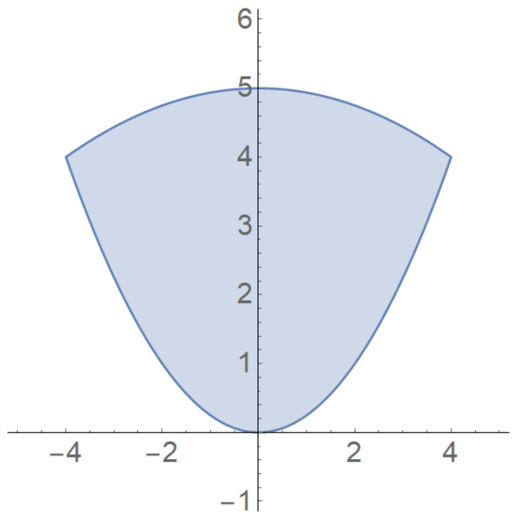
$$
\begin{aligned}
x^2 &= 4(c-b)^{-1}(b-\xi_1)(b-\xi_2)(b-\xi_3), \\
y^2 &= 4(b-c)^{-1}(c-\xi_1)(c-\xi_2)(c-\xi_3), \\
z &= \xi_1 + \xi_2 + \xi_3 - b - c,
\end{aligned}
$$

where $-\infty < \xi_1 < c < \xi_2 < b < \xi_3 < \infty$.

We consider the Helmholtz equation with a fixed boundary on a domain bounded by an upward opening elliptic paraboloid $\xi_1 = 0$ and a downward opening elliptic paraboloid $\xi_3 = 5$ for parameters $c = 1$ and $b = 3$:







Baer wave differential equations

The solution is $u = X_1(\xi_1)\, X_2(\xi_2)\, X_3(\xi_3)$, where X_1, X_2, X_3 satisfy the system of Baer wave DE

$$\boxed{(\xi_j - b)(\xi_j - c)\, X_j'' + \frac{1}{2}(2\xi_j - (b+c))\, X_j' + (\lambda + \mu\xi_j + \eta\xi_j^2)\, X_j = 0, \quad j = 1, 2, 3,}$$

for $0 < \xi_1 < c < \xi_2 < b < \xi_3 < 5$, where $\eta = \omega^2$.

Baer wave DE has regular singularities at b and c and irregular at infinity. The exponents at the regular singularities are 0 and $1/2$. The solution is

$$X_j(\xi_j) = (\xi_j - b)^{\rho/2}\, (\xi_j - c)^{\sigma/2}\, F_j(\xi_j),$$

where ρ and σ can be either 0 or 1.

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for $0 < \xi_1 < c < \xi_2 < b < \xi_3 < 5$, where $\eta = \omega^2$.

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$$X_j(\xi_j) = (\xi_j - b)^{\rho/2}\,(\xi_j - c)^{\sigma/2}\,F_j(\xi_j),$$

where ρ and σ can be either 0 or 1.

Theorem. *For each of the four possible configurations (σ, τ), the system of Baer wave DE has the Klein oscillation property, i.e., for each triple of nonnegative integers (m_1, m_2, m_3) there exists exactly one eigenvalue (λ, μ, η) such that the corresponding eigenfunctions $X_1(\xi_1)$, $X_2(\xi_2)$, $X_3(\xi_3)$ have exactly m_1 zeros on $(0, c)$, m_2 zeros on (c, b), and m_3 zeros on $(b, 5)$, respectively.*

Lowest eigenfrequencies

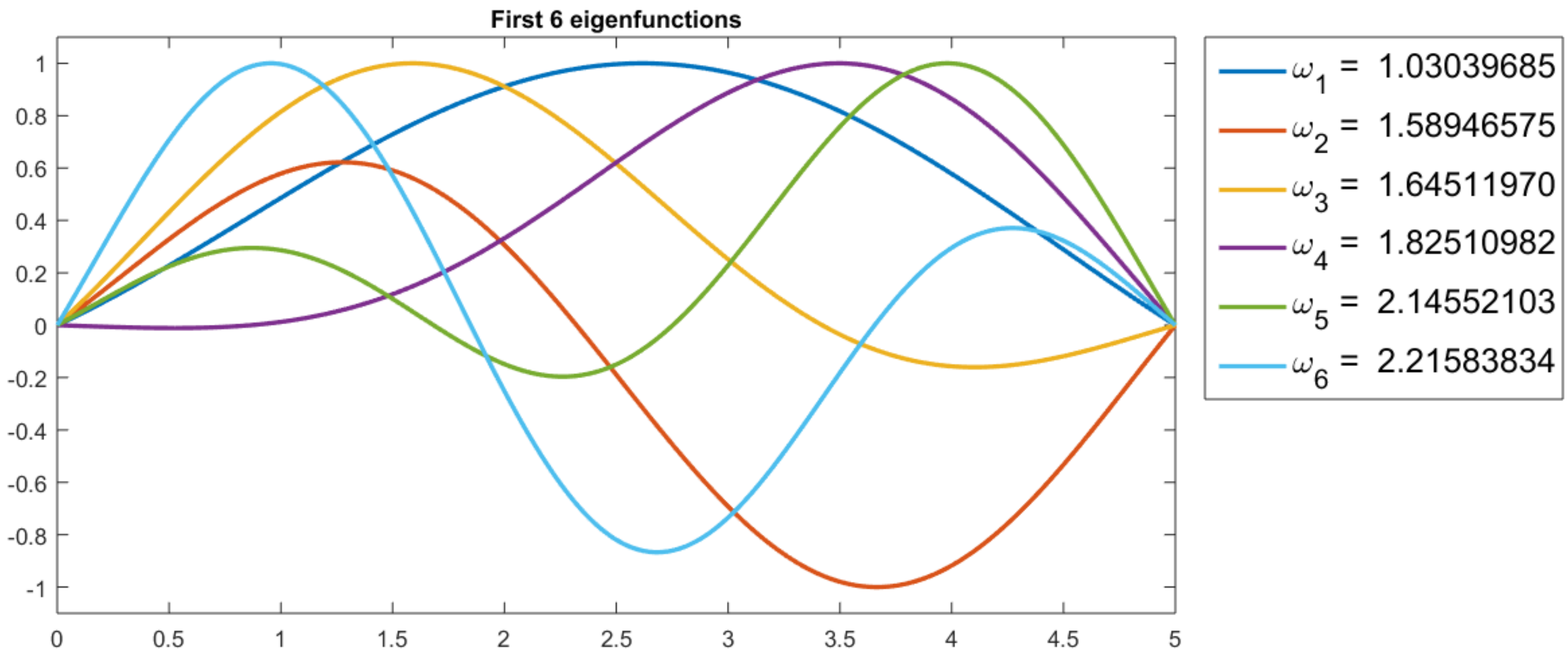
We combine the solutions $X_1(\xi_1)$, $X_2(\xi_2)$, $X_3(\xi_3)$ of a system of Baer wave DE

$$(\xi_j - 1)(\xi_j - 3)\, X_j'' + \frac{1}{2}(2\xi_j - 4)\, X_j' + (\lambda + \mu\xi_j + \eta\xi_j^2)\, X_j = 0, \quad j = 1, 2, 3,$$

in a smooth $X(\xi)$ bounded at singular points $\xi = 1$ in $\xi = 3$ that satisfies the DE

$$(\xi - 1)(\xi - 3)\, X'' + \frac{1}{2}(2\xi - 4)\, X' + (\lambda + \mu\xi + \eta\xi^2)\, X = 0$$

for $\xi \in [0, 5]$ subject to b.c. $X(0) = X(5) = 0$:



Example 2: ellipsoidal wave equation

Eigenmodes of a tri-axial ellipsoid Ω with semi-axes $z_0 > y_0 > x_0$ are solutions of

$$\nabla^2 u + \omega^2 u = 0 \quad \text{on} \quad \Omega = \left\{ (x,y,z) : \frac{x^2}{x_0^2} + \frac{y^2}{y_0^2} + \frac{z^2}{z_0^2} \le 1 \right\}, \quad u|_{\partial\Omega} = 0.$$

The solution in ellipsoidal coordinates is $u = X_1(\xi_1)\, X_2(\xi_2)\, X_3(\xi_3)$,

where $z_0 > \xi_1 > a > \xi_2 > b > \xi_3 > 0$ for $a = (z_0^2 - x_0^2)^{1/2}$ and $b = (z_0^2 - y_0^2)^{1/2}$.

After the substitution $t_i = \xi_i^2 / b^2$ we get a 3EP

$$\boxed{\begin{aligned} t_1(t_1-1)(t_1-c)\, X_1'' + \tfrac{1}{2}\big(3t_1^2 - 2\,(1+c)\,t_1 + c\big)\, X_1' + \big(\lambda + \mu\, t_1 + \eta\, t_1^2\big)\, X_1 &= 0, \\ t_2(t_2-1)(t_2-c)\, X_2'' + \tfrac{1}{2}\big(3t_2^2 - 2\,(1+c)\,t_2 + c\big)\, X_2' + \big(\lambda + \mu\, t_2 + \eta\, t_2^2\big)\, X_2 &= 0, \\ t_3(t_3-1)(t_3-c)\, X_3'' + \tfrac{1}{2}\big(3t_3^2 - 2\,(1+c)\,t_3 + c\big)\, X_3' + \big(\lambda + \mu\, t_3 + \eta\, t_3^2\big)\, X_3 &= 0, \end{aligned}}$$

where $X_i = X_i(t_i)$, $c = a^2/b^2$, $\omega^2 = 4\eta/b^2$, and

$$(z_0/b)^2 > t_1 > c > t_2 > 1 > t_3 > 0.$$

We can write $X_i(t_i) = t_i^{\rho/2}\,(t_i-1)^{\sigma/2}\,(t_i-c)^{\tau/2}\,F_i(t_i)$, where ρ, σ, and τ are either 0 or 1. This gives 8 possible types of ellipsoidal wave functions.

Numerical solution of ellipsoidal wave equation

This classical problem is difficult from computational point of view:

- (Levitina '99) a method that computes eigenvalue with a given multi-index.
- Graphs of first eigenfunctions are given by (Willatzen, Lew Yan Voon '05), who use the algorithm of (Arscott, Taylor, Zahar '83) to compute the first six eigenmodes.

MultiParEig computes first 50 eigenmodes accurately in couple of seconds:

- The Chebyshev collocation is used for discretization.
- We can use Jacobi-Davidson or subspace iteration to compute many low frequencies accurately and efficiently.
- The results agree to the results by Willatzen and Lew Yan Voon, but are much more accurate.

Conclusions

In several coordinate systems, when separation of variables is applied to PDEs (Helmholtz, Laplace, Schrödinger,...), we obtain 2EP or 3EP. This has rarely been used to solve such PDEs numerically, because:

- MEPs are less known,
- numerical methods for MEPs were not so efficient,
- discretization with finite differences gives MEPs with large matrices,
- lack of software.

This might change:

- spectral collocation and new numerical methods for MEPs can solve such problems both efficiently and accurately,
- methods are implemented in Matlab toolbox MultiParEig.

Some references

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Thank you for your attention!