fractional-Tikhonov and graph-Laplacian approximation

applied to signal and image restoration

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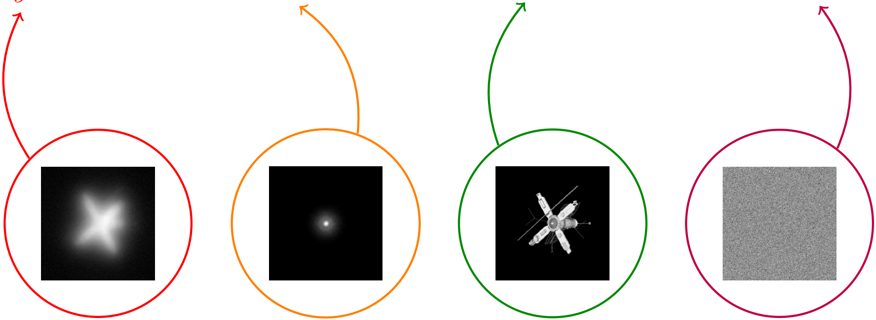
Our model problem

$$y^{\delta} \quad = \quad K \quad * \quad x \quad + \quad \text{noise}$$



- K represents the blur and it is severely ill-conditioned (compact integral operator of the first kind);
- y^{δ} are known measured data (blurred and noisy image);
- $\|\text{noise}\| \leq \delta$.

Filter based regularization methods

We substitute the $K^\dagger$ operator with a one-parameter family of continuous linear operators $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$,

$$K^\dagger y^\delta = \sum_{m:\,\sigma_m>0} \sigma_m^{-1}\langle y^\delta, u_m\rangle v_m$$

$$\Downarrow$$

$$R_\alpha y^\delta = \sum_{m:\,\sigma_m>0} F_\alpha(\sigma_m)\sigma_m^{-1}\langle y^\delta, u_m\rangle v_m$$

$\alpha = \alpha(\delta, y^\delta)$ is called rule choice.

Fractional Tikhonov filter functions

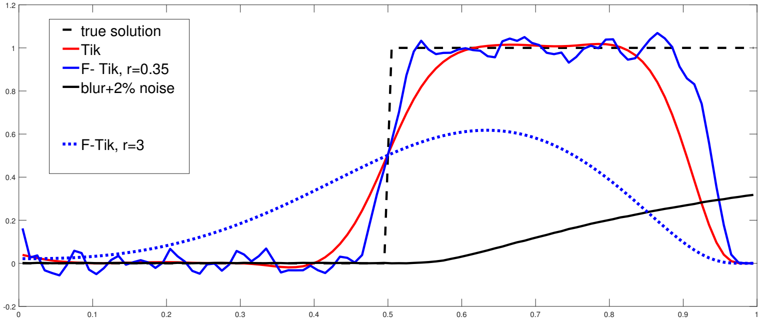
- Standard Tikhonov filter: $F_\alpha(\sigma_m) = \dfrac{\sigma_m^2}{\sigma_m^2 + \alpha}$, with $\alpha > 0$.
- Weighted/Fractional Tikhonov filter: $F_{\alpha,r}(\sigma_m) = \dfrac{\sigma_m^{r+1}}{\sigma_m^{r+1} + \alpha}$, with $\alpha > 0$ and $r \in [0, +\infty)$ (Hochstenbach and Reichel, 2011).

For $0 \leq r < 1$, fractional weighted filter smooths the reconstructed solution less than standard Tikhonov while for $r > 1$ it oversmooths.

An easy $1d$ example of oversmoothing - part 1

Blur taken from *Heat*(n, κ) in Regtools, $n = 100, \kappa = 1$ and 2% noise. True solution:

$$\mathbf{x}^\dagger : [0,1] \to \mathbb{R} \qquad \text{s.t.} \qquad \mathbf{x}^\dagger(t) = \begin{cases} 0 & \text{if } 0 \le t \le 0.5, \\ 1 & \text{if } 0.5 < t \le 1. \end{cases}$$



Let's reformulate the problem

- Tikhonov: $\underset{\mathbf{x}\in\mathbb{R}^n}{\operatorname{argmin}} \|K\mathbf{x} - \mathbf{y}\|_2^2 + \alpha\|\mathbf{x}\|_2^2$

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- Generalized Tikhonov: $\underset{\mathbf{x}\in\mathbb{R}^n}{\operatorname{argmin}} \|K\mathbf{x} - \mathbf{y}\|_2^2 + \alpha\|L\mathbf{x}\|_2^2$, with L semi-positive definite and $\ker(L) \cap \ker(K) = \vec{0}$. $\ker(L)$ should '*approximate the features* 'of $\mathbf{x}^\dagger$.

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Laplacian - Finite Difference approximation

Poisson (Sturm-Liouville) problem on $[0,1]$:

$$\begin{cases} -\Delta \mathbf{x}(t) = \mathbf{f}(t) & t \in (0,1), \\ \alpha_1 \mathbf{x}(0) + \beta_1 \mathbf{x}'(0) = \gamma_1, \\ \alpha_2 \mathbf{x}(1) + \beta_2 \mathbf{x}'(1) = \gamma_2. \end{cases}$$

If we consider Dirichlet homogeneous boundary conditions ($\mathbf{x}(0) = \mathbf{x}(1) = 0$) and 3-point stencil FD approximation:

$$-\Delta \mathbf{x}(t) \approx \frac{-\mathbf{x}(t-h) + 2\mathbf{x}(t) - \mathbf{x}(t+h)}{h^2}, \quad h^2 = n^{-2},$$

$$-L = \begin{bmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 2 \end{bmatrix} \qquad \ker(L) = \vec{0}.$$

Laplacian - Finite Difference approximation

$\cdots$

If we consider Neumann homogeneous boundary conditions ($\mathbf{x}'(0) = \mathbf{x}'(1) = 0$) and 3-point stencil FD approximation:

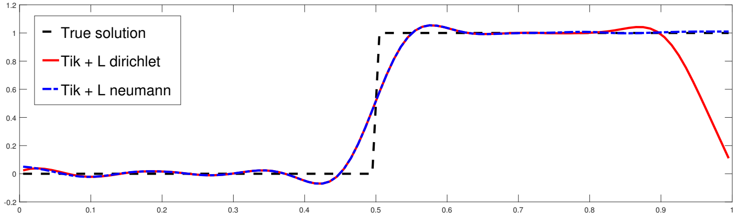
$$-\Delta \mathbf{x}(t) \approx \frac{-\mathbf{x}(t-h) + 2\mathbf{x}(t) - \mathbf{x}(t+h)}{h^2}, \quad h^2 = n^{-2},$$

$$-L = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 1 \end{bmatrix} \qquad \ker(L) = \mathsf{Span}\{\vec{1}\}.$$

An easy $1d$ example of oversmoothing - part 2

Blur taken from *Heat*(n, κ) in Regtools, $n = 100, \kappa = 1$ and 2% noise. True solution:

$$\mathbf{x}^\dagger : [0,1] \to \mathbb{R} \qquad \text{s.t.} \qquad \mathbf{x}^\dagger(t) = \begin{cases} 0 & \text{if } 0 \le t \le 0.5, \\ 1 & \text{if } 0.5 < t \le 1. \end{cases}$$



Graph Laplacian

- An image/signal $\mathbf{x}$ can be represented by a weighted undirected graph $\mathcal{G} = (V, E, w)$:
  - the nodes $v_i \in V$ are the pixels of the image/signal and $\mathbf{x}_i \geq 0$ is the color intensity of $\mathbf{x}$ at v_i.
  - an edge $e_{i,j} \in E \subseteq V \times V$ exists if the pixels v_i and v_j are connected, i.e., $v_i \sim v_j$.
  - $w : E \to \mathbb{R}$ is a similarity (positive) weight function, $w(e_{i,j}) = w_{i,j}$.
- The graph Laplacian is defined as

$$-\Delta_w^{(n)} \mathbf{x}_i = \sum_{v_j \sim v_i} w_{i,j} \left(\mathbf{x}_i - \mathbf{x}_j \right), \qquad \begin{cases} w_{i,j} > 0 & \text{if } v_j \sim v_i, \\ w_{i,j} = 0 & \text{otherwise.} \end{cases}$$

Graph Laplacian - Example

Example. In the $1d$ case, if we define

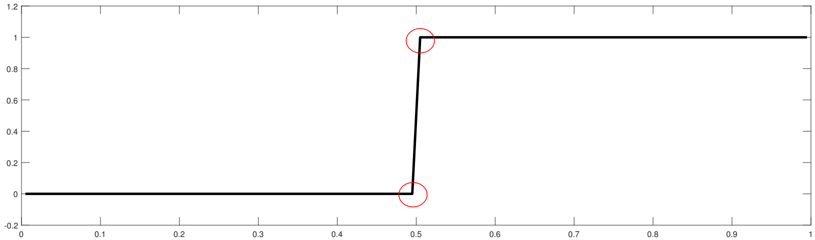
$$v_i \sim v_j \text{ iff } i = j+1 \text{ or } i = j-1, \qquad w_{i,j} = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise,} \end{cases}$$

then it holds

$$-\Delta_w^{(n)} = L_w^{(n)} = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 1 \end{bmatrix}.$$

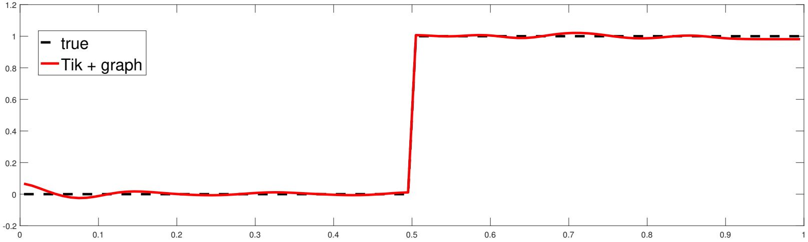
Question

Why should the red points be connected?



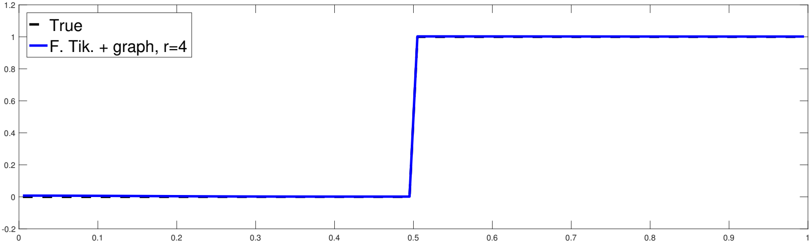
Answer

They should not, indeed



$$L_w^{(n)} = \begin{bmatrix} L_w^{(n/2)} & 0 \\ 0 & L_w^{(n/2)} \end{bmatrix}$$

Fractional Tikhonov + Graph Laplacian



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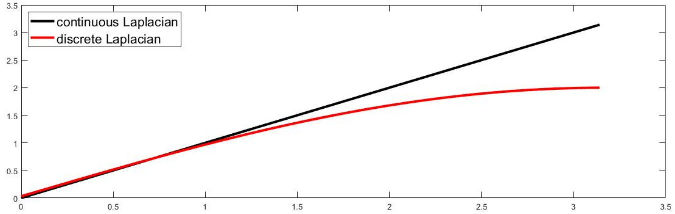
- Detection of the discontinuity points.
  - pre-denoising + first derivatives?
- SVD decomposition for the fractional-Tikhonov.
  - we need to bypass it finding a new smoothing parameter.
- Choice of the weights $w_{i,j}$.
  - finding the best weights that can approximate the Euclidean Laplacian.

Approximating the continuous Laplacian by graphs

$$-L_w^{(n)} = \begin{bmatrix} 1 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ & \ddots & \ddots & \ddots \\ & 0 & -1 & 1 \end{bmatrix} \Rightarrow \lambda_j(L_w^{(n)}) = 2 - 2\cos\left(\frac{(j-1)\pi}{n}\right)$$



FD approximation - 1/3

We have already highlighted that

Finite Difference 3-point stencil $\Longleftrightarrow$ graph-Laplacian

Can we argue the same relationship if we use a wider stencil, i.e., if we "connect" more points?

FD approximation - 2/3

Let us use a 5-point stencil.

$$-\Delta \mathbf{x}(t) \approx \frac{\mathbf{x}(t-2h) - 16\mathbf{x}(t-h) + 30\mathbf{x}(t) - 16\mathbf{x}(t+h) + \mathbf{x}(t-2h)}{12h^2},$$

then

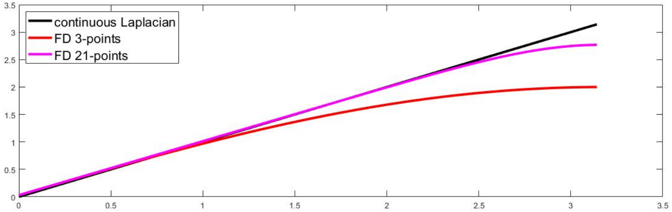
$$-L_w^{(n)} = -L_w^{(n)} = \begin{bmatrix} \frac{15}{12} & -\frac{16}{12} & \frac{1}{12} & 0 & \cdots & & \\ -\frac{16}{12} & \frac{31}{12} & -\frac{16}{12} & \frac{1}{12} & 0 & \cdots & \\ & \ddots & \ddots & \ddots & \ddots & & \\ \cdots & \frac{1}{12} & -\frac{16}{12} & \frac{30}{12} & -\frac{16}{12} & \frac{1}{12} & \cdots \\ & & & \ddots & \ddots & & \end{bmatrix}.$$

FD approximation 3/3

We have seen that negative weights appear. Does it make sense?

- Does it approximate better the continuous Laplacian? Yes.
- Is it still a graph-Laplacian? Yes.
- Does it improve the reconstruction of our signal/image? Yes.

Spectral approximation



graph-Laplacian - distributional point of view

$$
\begin{aligned}
-\mathbf{x}''(t_0) &= \langle \mathbf{x}''(t), \delta_{t_0}(t) \rangle \\
&= -\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \mathbf{x}''(t) \frac{\sin((t-t_0)\pi\epsilon^{-1})}{\pi(t-t_0)} \, d\mu(t) \\
&= -\lim_{\epsilon \to 0} \int_{-\infty}^{+\infty} \mathbf{x}(t) \left[\frac{\sin((t-t_0)\pi\epsilon^{-1})}{\pi(t-t_0)} \right]'' \, d\mu(t) \\
&\approx -\left(\alpha_{-k}\mathbf{x}(t_0 - kh) + \cdots + \alpha_0 \mathbf{x}(t_0) + \cdots + \alpha_k \mathbf{x}(t_0 - kh) \right),
\end{aligned}
$$

where

$$
\alpha_j = \mu\left(I_j\right) \cdot \left[\frac{\sin((t-t_0)\pi\epsilon^{-1})}{\pi(t-t_0)} \right]''_{|t=t_0+jh} .
$$

$[\alpha_{-k} \cdots \alpha_{-1} \quad \alpha_0 \quad \alpha_1 \cdots \alpha_k]$ is our stencil for the Toeplitz.

Signed measures - 1/2

$$\alpha_j = \mu\left(I_j\right) \cdot \left[\frac{\sin((t-t_0)\pi\epsilon^{-1})}{\pi(t-t_0)}\right]''_{|t=t_0+jh},$$

$$\begin{cases} \mu\left(I_j\right) > 0 & \text{always} \\ \left[\frac{\sin((t-t_0)\pi\epsilon^{-1})}{\pi(t-t_0)}\right]''_{|t=t_0+jh} & \text{changes sign.} \end{cases}$$

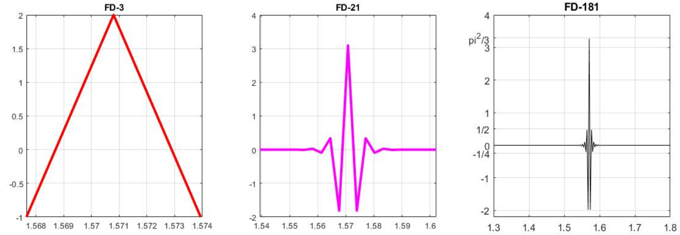
Is it so dramatic that the sequence α_j change signes?

Remark

The Lebesgue measure $d\mu(\cdot)$ on $[0,1]$ can be weakly approximated by signed measures: $d\mu(\cdot) = \lim_{n\to\infty} n^{-1}\frac{\sin(t\pi n)}{\pi t}d\mu(\cdot)$.

Signed measures - 2/2

Fact: the spectrum of the graph-Laplacian that arises from FD schemes with increasing connected points, converges to the spectrum of the continuous Laplacian operator.

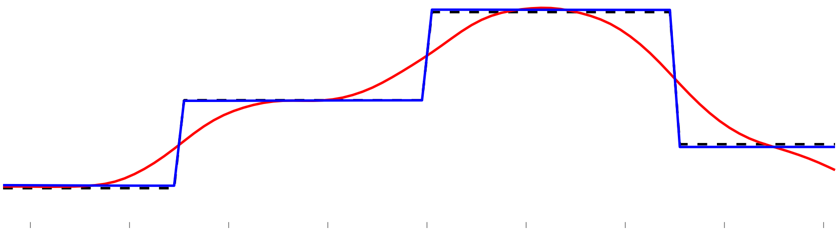


The stencil converges to the Fourier coefficients of $f(\theta) = \theta^2$:

$$\left[\cdots \quad -\frac{1}{4} \quad \frac{1}{2} \quad -2 \quad \frac{\pi^2}{3} \quad -2 \quad \frac{1}{2} \quad -\frac{1}{4} \cdots\right]$$

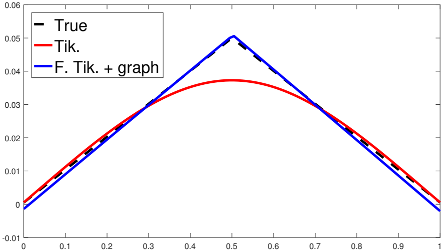
Example - heat$(n, 1)$ 5% noise

graph-Laplacian with 10 points connected.
No regularization parameters setting



$$L_w^{(n)} = \text{diag}\left(L_w^{(n/4)}, L_w^{(n/4)}, L_w^{(n/4)}, L_w^{(n/4)}\right)$$

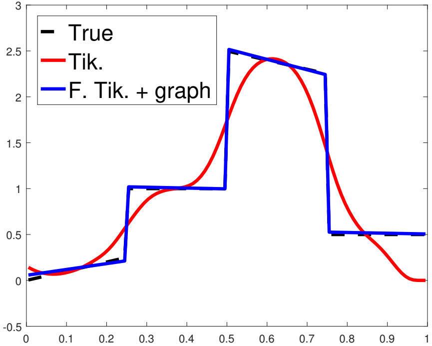
Example - deriv2$(n, 3)$, 2% noise



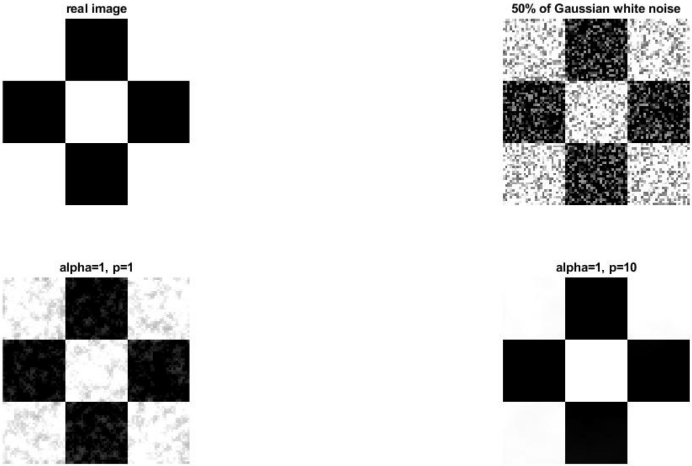
$$L_w^{(n)} = \begin{bmatrix} L_w^{(n/2)} & 0 \\ 0 & L_w^{(n/2)} \end{bmatrix} \quad L_w^{(n/2)} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

$$\ker(L_w^{(n/2)}) = \text{Span}\{\vec{1}, \vec{t}\}$$

Example - heat$(n, 1)$ 2% noise

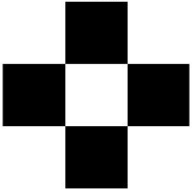


2D example - denosing 1/2

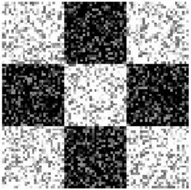


2D example - denoising 2/2

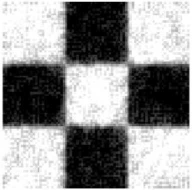
real image



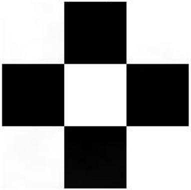
50% of Gaussian white noise



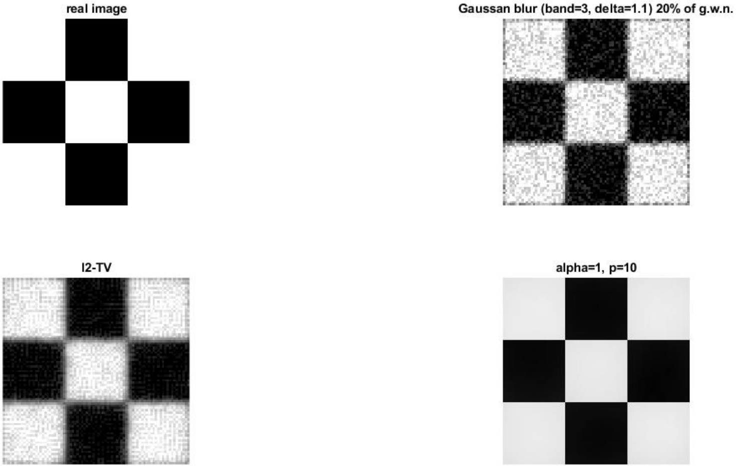
l2-TV



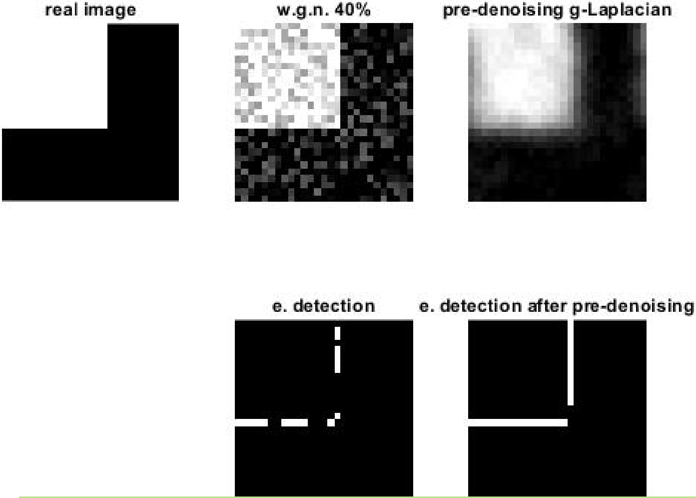
alpha=1, p=10



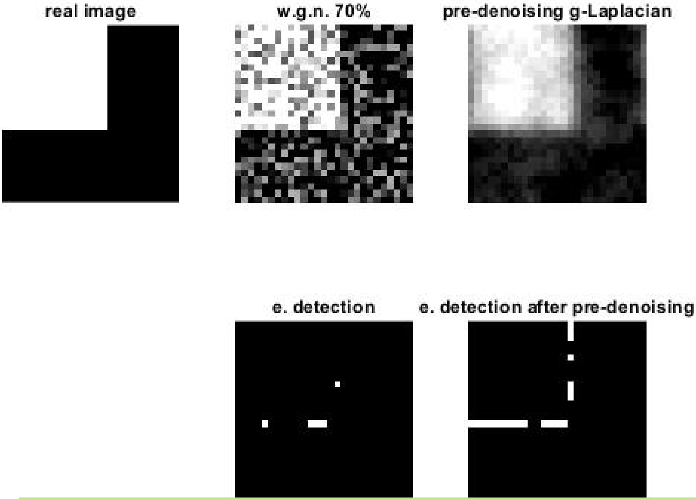
2D example - Gaussian blur



Edge detection - 1/2



Edge detection - 2/2



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