Linear autonomous systems partially ordered by the sharp partial order

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Outline of the talk

Linear autonomous systems

Linear autonomous system

A linear autonomous system can be given by

$$\begin{cases} x(k+1) = A\,x(k) \\ x(0) = x_0 \end{cases}$$

for $A \in \mathbb{R}^{n\times n}$ and $k = 0, 1, \ldots$.

Solution

The solution of a linear autonomous system is

$$x(k) = A^k x_0$$

Sharp partial ordering

Group inverse

Given a matrix $A \in \mathbb{R}^{n\times n}$, with index less than or equal to 1. The group inverse of A is the matrix $A^{\#}$ satisfying

$$AA^{\#}A = A, \quad A^{\#}AA^{\#} = A^{\#} \quad \text{and} \quad AA^{\#} = A^{\#}A$$

Sharp partial ordering

Given $A, B \in \mathbb{R}^{n\times n}$, with $ind(A) \leq 1$, $ind(B) \leq 1$,

$$A \overset{\#}{\leq} B \iff \begin{cases} AA^{\#} = BA^{\#} \\ A^{\#}A = A^{\#}B \end{cases}$$

where $A^{\#}$ is the group inverse of A.

Properties

Invariance under similarities

Given $A, B \in \mathbb{R}^{n\times n}$, with $\mathit{ind}(A) \leq 1$ and $\mathit{ind}(B) \leq 1$, and $P \in \mathbb{R}^{n\times n}$ nonsingular,

$$A \overset{\#}{\leq} B \quad \Longrightarrow \quad PAP^{-1} \overset{\#}{\leq} PBP^{-1}$$

Spectrum

Given $A, B \in \mathbb{R}^{n\times n}$ such that $A \overset{\#}{\leq} B$, then

$$\mathit{Spectrum}(A) \subseteq \mathit{Spectrum}(B) \cup \{0\}.$$

Characterizations of the sharp partial ordering

First characterization

Given $A, B \in \mathbb{R}^{n\times n}$, with $ind(A) \leq 1$ and $ind(B) \leq 1$, the following statementes are equivalent:

- $A \overset{\#}{\leq} B$
- $A^2 = BA = AB$
- There exist nonsingular matrices P, C_1 and C_2 such that

$$\underbrace{A = P \begin{bmatrix} C_1 & & \\ & O & \\ & & O \end{bmatrix} P^{-1}}_{\text{Core-nilpotent decomposition}} \quad and \quad B = P \begin{bmatrix} C_1 & & \\ & C_2 & \\ & & O \end{bmatrix} P^{-1}$$

Characterizations of the sharp partial ordering

Second characterization

Given $A, B \in \mathbb{R}^{n\times n}$, with $ind(A) \leq 1$ and $ind(B) \leq 1$, the following statementes are equivalent:

- $A \overset{\#}{\leq} B$
- There exist an idempotent matrix Q such that

$$A = QB = BQ$$

Characterizations of the sharp partial ordering

Third characterization

Given $A, B \in \mathbb{R}^{n\times n}$, with $ind(A) \leq 1$ and $ind(B) \leq 1$, the following statementes are equivalent:

- $A \overset{\#}{\leq} B$
- There exist an idempotent matrix T such that

$$B = U \begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix} U^T \quad and \quad A = U \begin{bmatrix} T\Sigma K & T\Sigma L \\ O & O \end{bmatrix} U^T$$

  where

  - U is an orthogonal matrix,
  - Σ is a diagonal definite positive matrix,
  - K and L matrices such that $KK^T + LL^T = I$,

  } Hartwig Spindelböck decomposition

  - $T\Sigma K = \Sigma K T$.

Sharp partial ordered autonomous systems

Definition

Consider the autonomous systems

$$(1) \begin{cases} x(k+1) = A\, x(k) \\ x(0) = x_0 \end{cases} \quad \textit{and} \quad (2) \begin{cases} \bar{x}(k+1) = B\, \bar{x}(k) \\ \bar{x}(0) = \bar{x}_0 \end{cases}$$

for $k = 0, 1, \ldots$ and $A, B \in \mathbb{R}^{n \times n}$ having index 1.

The systems are ordered under the sharp partial order if $A \overset{\#}{\leq} B$.

- The system (1) is a predecessor of system (2) under the sharp partial order
- The system (2) is a successor of system (1) under the sharp partial order

Solution of two ordered autonomous systems

First characterization: $A \overset{\#}{\leq} B$

$$A = P \begin{bmatrix} C_1 & & \\ & O & \\ & & O \end{bmatrix} P^{-1} \quad \textit{and} \quad B = A + \underbrace{P \begin{bmatrix} O & & \\ & C_2 & \\ & & O \end{bmatrix} P^{-1}}_{\Gamma}$$

Theorem

Let $A, B \in \mathbb{R}^{n \times n}$ be the state matrices of two sharp partial ordered autonomous systems.
The solutions of both systems are related by

$$\bar{x}(k) = x(k) + \Gamma^k x_0$$

provided that $\bar{x}(0) = x(0) = x_0$.

Solution of two ordered autonomous systems

Difference between both solutions

Let $A, B \in \mathbb{R}^{n \times n}$ be the state matrices of two sharp partial ordered autonomous systems. Then,

$$\|\bar{x}(k) - x(k)\| \leq \|C_2\|_F^k \|x_0\|$$

where $\|\cdot\|$ is compatible with the $P-$matrix norm.

Remarks

- System (2) can be seen as a perturbation of System (1): $B = A + \Gamma$.
- If the perturbation C_2 is "small" then $\bar{x}(k)$ is close to $x(k)$.
- Since $\sigma(B) = \sigma(A) \cup \sigma(C_2)$, the stability of system (2) depends on stability of system (1) and on the perturbation.

Solution of two ordered autonomous systems

Algorithm

Inputs: The matrix A of index at most 1, the initial condition x_0, and the nonzero perturbation numbers $\varepsilon_1, \ldots, \varepsilon_\ell$.
Outputs: The matrix B and the solutions $x(k)$ and $\bar{x}(k)$.

1. Compute the core-nilpotent decomposition of A:
$$A = P \begin{bmatrix} C_1 & & \\ & O & \\ & & O \end{bmatrix} P^{-1}.$$
2. Select $C_2 = diag(\varepsilon_1, \ldots, \varepsilon_\ell)$.
3. Construct $\Gamma = P \begin{bmatrix} O & & \\ & C_2 & \\ & & O \end{bmatrix} P^{-1}$ and $B = A + \Gamma$.
4. The solutions are: $x(k) = A^k x_0$ and $\bar{x}(k) = x(k) + \Gamma^k x_0$.

Solution of two ordered autonomous systems

Example: $x(k+1) = A\,x(k)$ and $\bar{x}(k+1) = B\,\bar{x}(k)$

Let

$$A = \begin{bmatrix} 0.1739 & -0.0607 & -0.0662 & 0.0763 & 0.0596 & -0.0400 \\ 0.4338 & -0.0725 & -0.1948 & 0.2847 & 0.1164 & -0.0720 \\ -0.8839 & 0.3844 & 0.3080 & -0.2966 & -0.3343 & 0.2302 \\ -0.0239 & 0.1583 & -0.0471 & 0.1691 & -0.0697 & 0.0584 \\ 0.1986 & -0.1166 & -0.0579 & 0.0304 & 0.0875 & -0.0624 \\ -0.5959 & 0.2936 & 0.1947 & -0.1587 & -0.2395 & 0.1674 \end{bmatrix} =$$

$$= P \left[\begin{array}{cc|cc} 0.5000 & -0.3333 & & \\ 0 & 0.3333 & & \\ \hline & & O & \\ & & & O \end{array}\right] P^{-1}$$

and

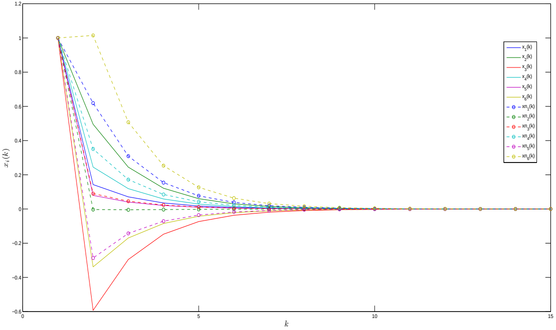
$$B = P \left[\begin{array}{cc|cc|c} 0.5000 & -0.3333 & & & \\ 0 & 0.3333 & & & \\ \hline & & 0.5000 & 0 & \\ & & 0 & 0.5000 & \\ \hline & & & & O \end{array}\right] P^{-1}$$

Solution of two ordered autonomous systems

Then

$$\|\bar{x}(k) - x(k)\| \leq \|C_2\|_F^k \|x_0\| = 0.7071\|x_0\|.$$

Figure: Evolution of the 15th first iterations of $x_i(k)$ and $\bar{x}_i(k)$



Solution of two ordered autonomous systems

Second characterization: $A \overset{\#}{\leq} B$

$$A = QB$$

where Q is an idempotent matrix.

Theorem

Let $A, B \in \mathbb{R}^{n\times n}$ be the state matrices of two sharp partial ordered autonomous systems.
The solutions of both systems are related by

$$x(k) = Q\bar{x}(k)$$

provided that $\bar{x}(0) = x(0) = x_0$.

Solution of two ordered autonomous systems

Remarks

- Solution of System (1) is a projection of the solution of System (2): $x(k) = Q\bar{x}(k)$.
- All the idempotent matrices Q_Z satisfying $A = Q_Z B = BQ_Z$ are

$$Q_Z = P \begin{bmatrix} I & & \\ & O & \\ & & Z \end{bmatrix} P^{-1}$$

  In particular $Z = O \Rightarrow Q_Z = AA^\# = A^\# A$.
- Both solutions are as close as the magnitude of the matrix $Q - I$:

$$\|x(k) - \bar{x}(k)\| \leq \|Q - I\|_F \|x_0\|$$

  where $\|\cdot\|_F$ is the Frobenius norm.

Solution of two ordered autonomous systems

Third characterization: $A \overset{\#}{\leq} B$

$$B = U \begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix} U^T \quad \text{and} \quad A = \underbrace{U \begin{bmatrix} T & O \\ O & I \end{bmatrix} U^T}_{\widetilde{\Gamma}} B$$

Theorem

Let $A, B \in \mathbb{R}^{n \times n}$ be the state matrices of two sharp partial ordered autonomous systems.
The solutions of both systems are related by

$$x(k) = \widetilde{\Gamma} \bar{x}(k)$$

provided that $\bar{x}(0) = x(0) = x_0$.

Solution of two ordered autonomous systems

Difference between both solutions

Let $A, B \in \mathbb{R}^{n\times n}$ be the state matrices of two sharp partial ordered autonomous systems. Then,

$$\|x(k) - \bar{x}(k)\| \leq \|T - I\|_F \|\Sigma\|_F^k \|x_0\|$$

.

Remarks

- Solution of System (1) is a projection of the solution of System (2): $x(k) = \widetilde{\Gamma}\bar{x}(k)$.
- Both solutions are as close as the magnitude of the singular value matrix Σ.
- Since $\sigma(T\Sigma K) \subseteq \sigma(\Sigma K)$, the stability of System (2) implies the stability of System (1).

Solution of two ordered autonomous systems

Algorithm

Inputs: The matrix B of index at most 1 and the initial condition x_0.
Outputs: The matrix A and the solutions $\bar{x}(k)$ and $x(k)$.

1. Compute the SVD of B: $B = USV^T$ and $r = rank(B)$.
2. Assign to Σ the first r rows and the first r columns of S.
3. Compute $M = SV^T U$.
4. Assign to $\widetilde{M}$ the first r rows and the first r columns of M.
5. Compute $R = \Sigma^{-1}\widetilde{M}$.
6. Assign to K the first r rows and the first r columns of R.
7. Assign to L the first r rows and the last $n-r$ columns of R.

Solution of two ordered autonomous systems

Algorithm

Until here we have constructed the Hartwig-Spindelböck decomposition of B:

$$B = U \begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix} U^T$$

8. Find a matrix T such that $\Sigma K T = T \Sigma K$ and $T^2 = T$.
9. Construct

$$A = U \begin{bmatrix} T\Sigma K & T\Sigma L \\ O & O \end{bmatrix} U^T \quad \text{and} \quad \widetilde{\Gamma} = U \begin{bmatrix} T & O \\ O & I \end{bmatrix} U^T.$$

10. The solutions are: $\bar{x}(k) = B^k x_0$ and $x(k) = \widetilde{\Gamma}\bar{x}(k)$.

Solution of two ordered autonomous systems

Example: $\bar{x}(k+1) = B\ \bar{x}(k)$ and $x(k+1) = A\ x(k)$

Let

$$B = \begin{bmatrix} 0.3748 & -0.0548 & -0.1024 & 0.0447 & 0.2241 & 0.1326 \\ -0.2717 & 0.4732 & -0.2945 & -0.1694 & 0.0093 & 0.2499 \\ -0.3656 & 0.1359 & 0.3203 & -0.1150 & -0.1290 & 0.2435 \\ 0.2810 & -0.1305 & 0.0172 & 0.4182 & -0.0674 & -0.1675 \\ -0.2640 & 0.2224 & -0.1157 & -0.2485 & 0.0016 & 0.1178 \\ 0.3813 & -0.1425 & 0.2049 & 0.1514 & 0.1744 & 0.2454 \end{bmatrix}$$

Then

$$\Sigma = \begin{bmatrix} 1.0819 & 0 & 0 & 0 \\ 0 & 0.5784 & 0 & 0 \\ 0 & 0 & 0.5256 & 0 \\ 0 & 0 & 0 & 0.3085 \end{bmatrix}.$$

Choose

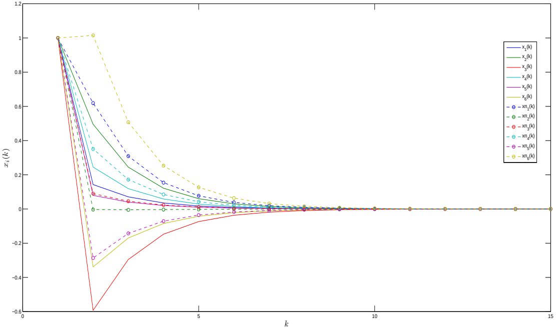
$$T = \begin{bmatrix} -0.1860 & -0.1428 & 0.0821 & 0.2657 \\ 0.6177 & 0.5894 & -0.3608 & -0.0162 \\ -1.0127 & -0.9133 & 0.5509 & 0.4261 \\ -0.1855 & -0.0384 & 0.0021 & 1.0455 \end{bmatrix}.$$

Solution of two ordered autonomous systems

Then

$$\|x(k) - \bar{x}(k)\| \leq \|T - I\|_F^k \|\Sigma\|_F^k \|x_0\|.$$

Figure: Evolution of the 15th first iterations of $x_i(k)$ and $\bar{x}_i(k)$



Conclusions

- We have introduced the concept of sharp partial ordered autonomous systems.
- The successor system can be seen as a perturbation of its predecessor. The difference between their solutions is given by the magnitude of the perturbation matrix C_2.
- The solution of the predecessor system can be obtained as a projection of the solution of its successor system and, in this case, the difference between the solutions is given by the magnitude of the singular value matrix Σ.

Future work

- Define ordered autonomous systems for other matrix partial orders like minus, cn, star ...
- Extend the concept of sharp partial ordered autonomous systems to linear control systems.
- Study linear control systems for different orders.

THANK YOU VERY MUCH FOR

YOUR ATTENTION

Solution of $T\Sigma K = \Sigma K T$ and $T^2 = T$

Proposition

Let Σ and K be the matrices of the Hartwig-Spindelböck decomposition of B.
There exists a nontrivial idempotent matrix T such that $\Sigma K T = T \Sigma K$

$$\Updownarrow$$

There exists a nonsingular matrix S such that $\Sigma K = S \begin{bmatrix} S_1 & O \\ O & S_2 \end{bmatrix} S^{-1}$

Remarks

- The matrix T can always be constructed since the block partition of ΣK can always be done using its Jordan canonical form.
- If ΣK is diagonalizable, we can construct several matrices T with different rank by choosing adequate blocks in ΣK.
- Alternatively we can select T using the Schur decomposition of ΣK.