A fast algorithm for computing the mock-Chebyshev interpolation

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Problem statement

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Linear interpolation:

- nodes:

$$x_0, \dots, x_n, \quad x_i \in [-1, 1]$$

- data:

$$f_0, \dots, f_n, \quad f_i \in \mathbb{C}$$

- basis functions:

$$\phi_0(x), \dots, \phi_n(x)$$

Interpolation conditions:

$$\sum_{i=0}^{n} a_i \phi_i(x_j) = f_j, \quad j = 0, \dots, n$$

Polynomial interpolation

Problem statement

Given:

$$(x_0, f_0), \dots, (x_n, f_n), \quad x_i \in [-1, 1]$$

Compute:

$$p_n(x) = \sum_{i=0}^{n} a_i x^i, \quad p_n(x_i) = f_i$$

Unisolvence:

$$x_i \neq x_j, \qquad i \neq j$$

Interpolation error:

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i), \quad \xi \in \,]-1, 1[$$

Polynomial interpolation

Lebesgue constant

Lagrange form:

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$$

$$\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, \dots, n$$

$$\ell_i(x_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

Lebesgue function:

$$L_n(x) = \sum_{i=0}^{n} |\ell_i(x)|$$

Lebesgue constant:

$$\Lambda_n(x_0, \dots, x_n) = \max_{x\in[-1,1]} L_n(x)$$

Polynomial interpolation

Error estimates

- Error formula 1:

$$\|f - p_n\|_\infty \leq \max_{x\in[-1,1]} \left(\frac{|f^{(n+1)}(x)|}{(n+1)!} \right) \max_{x\in[-1,1]} \prod_{j=0}^{n} |x - x_j|$$

- Error formula 2:

$$\|f - p_n\|_\infty \leq (1 + \Lambda_n) \max_{x\in[-1,1]} |f(x) - p_n^*(x)|,$$

$$\Lambda_n = \max_{x\in[-1,1]} \sum_{i=0}^{n} |\ell_i(x)|, \quad \ell_i(x) = \frac{\prod_{j=0,\ j\neq i}^{n}(x - x_j)}{\prod_{j=0,\ j\neq i}^{n}(x_i - x_j)}$$

$p_n^* \in \Pi_n$ is the best uniform polynomial approximation to f

$\Rightarrow$ Small Lebesgue constant implies $\|f - p_n\|_\infty \approx \|f - p_n^*\|_\infty$

Polynomial interpolation

Optimize:

$$\min_{\{x_0,\dots,x_n\}\subset[-1,1]} \Lambda_n(x_0,\dots,x_n)$$

Solution:

$$\Lambda_n^{\text{Optimal}} = \frac{2}{\pi}\ln(n+1) + 0.5212\ldots + o(1) \qquad \text{[Vértesi'90]}$$

- set T: Chebyshev points in $(-1,1)$

$$x_i = -\cos\left(\frac{\pi(2i+1)}{2(n+1)}\right), \quad i = 0,\dots,n,$$

$$\Lambda_n(T) = \frac{2}{\pi}\ln(n+1) + 0.9625\ldots + o(1)$$

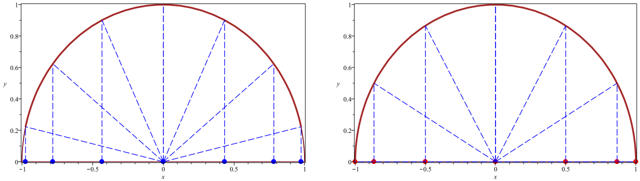
- set C: Chebyshev-Lobatto points on $[-1,1]$

$$x_i = -\cos\left(\frac{i\pi}{n}\right), \quad i = 0,\dots,n,$$

$$\Lambda_n(C) = \begin{cases} \Lambda_{n-1}(T), & n \text{ odd} \\ \Lambda_{n-1}(T) - \alpha_n, & 0 < \alpha_n < \frac{1}{n^2}, \quad n \text{ even} \end{cases}$$

Polynomial interpolation

Interpolation points

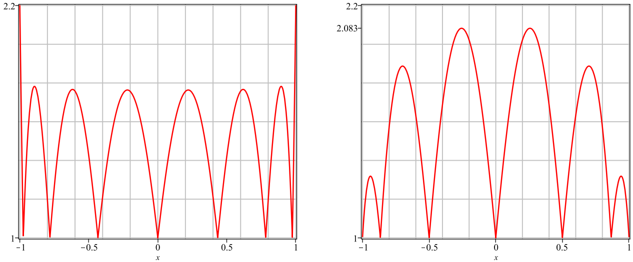


7 Chebyshev (•) and Chebyshev-Lobatto (•) points, $n = 6$

Polynomial interpolation

Lebesgue function

$$L_n(x) = \sum_{i=0}^{n} |\ell_i(x)|, \quad -1 \le x \le 1$$



Lebesgue function for Chebyshev and for Chebyshev-Lobatto points, $n + 1 = 7$

Mock-Chebyshev interpolation

When the data $f_0, \dots, f_n$ are provided at equidistant points only

Polynomial interpolation with equidistant points is unreliable:

- it is severely ill-conditioned ($\Lambda_n \to \infty$ exponentially fast)
  set E: equidistant points

$$x_i = -1 + \frac{2i}{n}, \quad i = 0, \dots, n,$$

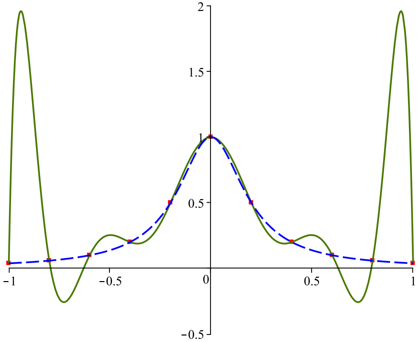
$$\Lambda_n(E) = \frac{2^{n+1}}{e\,n(\ln(n) + \gamma)} + o(1)$$

- it leads to Runge's phenomenon

  The Runge phenomenon: polynomial interpolation of a function f, using equidistant interpolation points on $[-1, 1]$, could diverge on parts of this interval even if f is analytic everywhere on the interval.

Mock-Chebyshev interpolation

Runge's phenomenon



Graphs of $f(x) = \frac{1}{1+25x^2}$ (dashed line) and its polynomial interpolant (solid line) at 11 equidistant points (solid boxes), $-1 \le x \le 1$

Mock-Chebyshev interpolation

When the data $f_0, \dots, f_n$ are provided at equidistant points only

Among all methods that have been proposed to defeat Runge's phenomenon, the mock-Chebyshev interpolation algorithm is the exception

- being blessed with the same $\mathcal{O}(1)$ condition method as standard Chebyshev interpolation with n points (w.r.t. numerical ill-conditioning)
- as it falls under the umbrella of standard Chebyshev interpolation (w.r.t. lack of a rigorous theory)

The mock-Chebyshev grid: a subset of $(n+1)$ points from an equispaced grid with $\mathcal{O}(n^2)$ points chosen to mimic the non-uniform $n+1$-point Chebyshev-Lobatto grid [Boyd, Xu, 2009].

Mock-Chebyshev interpolation

The Mock-Chebyshev Grid [Boyd, Xu, 2009]

$$\tilde{x}_j = \bar{x}_l, \quad \text{such that } |x_j - \bar{x}_l| = \min_k |x_j - \bar{x}_k|,$$
$$\forall k = 0, 1, \ldots, m, \quad j = 0, 1, \ldots, n, \quad l \in \{0, 1, \ldots, m\},$$

- x_j: points of Chebyshev-Lobatto grid with $(n+1)$ points

$$x_j = -\cos\left(\frac{j\pi}{n}\right), \quad j = 0, 1, \ldots, n$$

- $\bar{x}_k$: points of equispaced grid with $m = \mathcal{O}(n^2)$ points

$$\bar{x}_k = -1 + \frac{2k}{m}, \quad k = 0, 1, \ldots, m$$

- $\tilde{x}_j$: subset of $(n+1)$ points, imitating Chebyshev-Lobatto grid, selected from equispaced grid with $m = \mathcal{O}(n^2)$ points

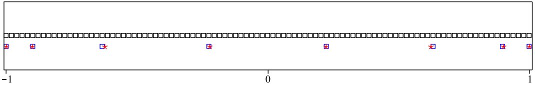
Mock-Chebyshev interpolation

The Mock-Chebyshev Grid [Boyd, Xu, 2009]

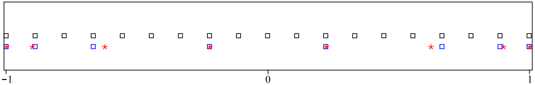
- To avoid the fact that the endpoints −1 and 1 can be selected more than once, m is greater than $\frac{2n^2}{\pi^2}$
- The computational cost is $\mathcal{O}(nm)$

Example: 8 mock-Chebyshev points for $x_j = -\cos(j\pi/7), \quad j = 0, 1, \ldots, 7$

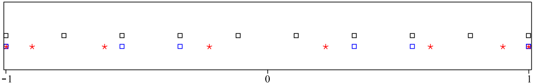
- $m = 99$



- $m = 18$



- for $m = 9$, the endpoints −1 and 1 repeat!



Mock-Chebyshev interpolation

Mock-X points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ on $[a, b]$

Given:

a set of non-uniform $(n+1)$ points

$$X_n = \{x_j : j = 0, 1, \ldots, n\}, a = x_0 < x_1 < \cdots < x_n = b$$

Find:

a subset of uniformly distributed $(n+1)$ points

$$\tilde{X}_n = \{\tilde{x}_j : j = 0, 1, \ldots, n\}, a = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_n = b,$$

in the sense that the subset $\tilde{X}_n$ of $(n+1)$ points mimic the behavior of the given non-uniform $(n+1)$ points

Mock-Chebyshev interpolation

A new approach for computing the mock-X points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$

Here we give a new method for generating the mock-X points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$:

$$\tilde{x}_j = \tilde{x}_0 + S_j \tilde{h}, \quad \tilde{h} = \frac{2}{S_n}, \quad \tilde{x}_0 = x_0, \quad S_0 = 0, \quad j = 1, 2, \ldots, n,$$

$$S_j = \left\lceil \frac{h_j(X_n)}{\min h_j(X_n)} \right\rceil + S_{j-1}, \quad h_j(X_n) = |x_j - x_{j-1}|, \quad j = 1, 2, \ldots, n$$

- where $\lceil \cdot \rceil$ denotes the ceil function. For computing S_j, we can also use the floor function $\lfloor \cdot \rfloor$ or the round function $\mathtt{round}(\cdot)$
- the points $\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n$ are a subset of $(S_n + 1)$ equispaced points with gap $\tilde{h} = 2/S_n$

Mock-Chebyshev interpolation

The mock-Chebyshev grid with our method

When we take a set of $(n+1)$ Chebyshev-Lobatto points as X_n in our method, then we obtain the set of mock-Chebyshev points.

For this case, let $\bar{X}_m = \{\bar{x}_k = -1 + 2k/m, \quad k = 0, 1, \ldots, m\}$ be the superset of $\tilde{X}_n$. Then we have the following properties:

- $\min h_j(X_n) = h_1(X_n) = h_n(X_n), \quad j = 1, 2, \ldots, n$
- $m > \frac{4n^2}{\pi^2}$, therefore the distance between the points $\bar{x}_0, \bar{x}_1, \cdots, \ \bar{x}_m$ ensures that the endpoints -1 and 1 never repeat
- $\tilde{x}_j$ are symmetric with respect to the origin

Mock-Chebyshev interpolation

The mock-Chebyshev grid with our method

- For $n \leq 2i+1, \quad i = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$,

$$\frac{x_i + x_{i+1}}{2} < -1 + \frac{4(x_{i+1}+1)}{(n-2)x_1 + n + 2} \leq \tilde{x}_{i+1},$$

$$\tilde{x}_{i+1} \leq x_{i+1} + \frac{i}{2}(x_1 + 1) < \frac{x_{i+1} + x_{i+2}}{2}$$

By using the symmetry, we conclude:

$$\frac{x_i + x_{i+1}}{2} < \tilde{x}_{i+1} < \frac{x_{i+1} + x_{i+2}}{2}, \quad i = 0, 1, \ldots, n-2$$

Mock-Chebyshev interpolation

The mock-Chebyshev grid with our method

- The distance between the points $\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_m$ ensures that a grid point $\bar{x}_k$ nearest to a Chebyshev-Lobatto point x_j is never repeated. Moreover, numerical results suggest that

$$\max_{0 \le j \le n} |x_j - \tilde{x}_j| = O((n \ln n)^{-1})$$

- The computational cost is $\mathcal{O}(n)$ (Compare this with $\mathcal{O}(nm)$)

Mock-Chebyshev interpolation

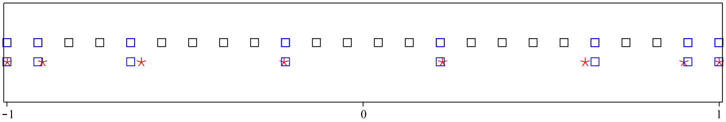
Example: 8 mock-Chebyshev points with our method

$$X_7 = \left\{x_j = -\cos\left(\frac{j\pi}{7}\right), \quad j = 0, 1, \ldots, 7\right\}$$

$$\tilde{X}_7 = \left\{-1, -\frac{21}{23}, -\frac{15}{23}, -\frac{5}{23}, \frac{5}{23}, \frac{15}{23}, \frac{21}{23}, 1\right\}$$

$$\bar{X}_{23} = \left\{\bar{x}_k = -1 + \frac{2k}{23}, \quad k = 0, 1, \ldots, 23\right\}$$

$$\tilde{X}_7 \subset \bar{X}_{23}$$

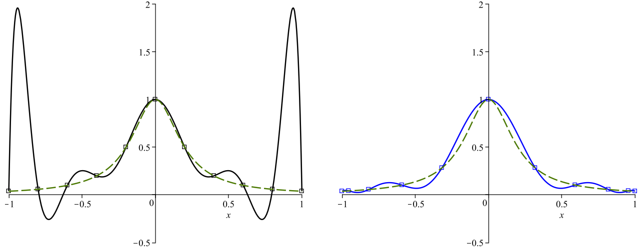


Graph of 8 Chebyshev-Lobatto points ($*$) with their mock-Chebyshev points ($\square$) with gap 2/23 for degree $n = 7$

Mock-Chebyshev interpolation

Mock-Chebyshev interpolation for Runge function, $n = 10$

$$f(x) = \frac{1}{1+25x^2}, \quad -1 \le x \le 1$$

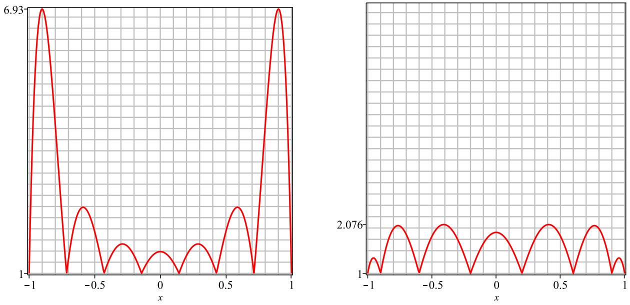


Polynomial interpolation with equidistant (□) and with mock-Chebyshev points (□)

Mock-Chebyshev interpolation

Lebesgue function

$$L_n(x) = \sum_{i=0}^{n} |\ell_i(x)|, \quad -1 \le x \le 1$$



Lebesgue function for equidistant and for mock-Chebyshev points, $n = 7$

Mock-Chebyshev interpolation

Lebesgue constants

Lebesgue constants of Chebyshev-Lobatto points and mock-Chebyshev points on $[-1, 1]$

n	Chebyshev-Lobatto points	mock-Chebyshev points
5	1.99	2.25
10	2.42	2.58
20	2.87	2.87
40	3.31	3.33
100	3.89	3.80

Bivariate polynomial interpolation

Problem statement

Given: (x_i, y_i, f_i), $i = 0, \ldots, N$, $\quad (x_i, y_i) \in [-1, 1]^2$

$$N + 1 = \dim\left(\Pi_n^2\right) = \frac{(n+1)(n+2)}{2}$$

Find:

$$p_n(x, y) = \sum_{k_1+k_2=0}^{n} c_{k_1 k_2} x^{k_1} y^{k_2}, \quad p_n(x_i, y_i) = f_i$$

Bivariate polynomial interpolation

Problem statement

Unisolvence: $\det(\tau_n) \neq 0$

$$\mathbf{x} = (x, y) \in \{\mathbf{x} \,:\, \|\mathbf{x}\|_\infty \le 1\}, \quad \mathbf{x}_i = (x_i, y_i), \quad i = 0, \dots, N$$

$$\varphi_n(x, y) = \left(x^0 y^0, \dots, x^{j_1} y^{j_2}, \dots, x^0 y^n\right), \quad 0 \le j_1 + j_2 \le n$$

$$\tau_n(\mathbf{x}_0, \dots, \mathbf{x}_N) = \begin{bmatrix} \varphi_n(x_0, y_0) \\ \vdots \\ \varphi_n(x_N, y_N) \end{bmatrix}$$

$$\ell_i(\mathbf{x}) = \frac{\det(\tau_n(\mathbf{x}_0, \dots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_N))}{\det(\tau_n(\mathbf{x}_0, \dots, \mathbf{x}_N))}$$

$$p_n(x, y) = \sum_{i=0}^{N} f_i \ell_i(\mathbf{x}), \quad \det(\tau_n) \neq 0$$

Bivariate polynomial interpolation

Problem statement

Error estimate:

$$\max_{\|(x,y)\|_\infty \le 1} |f(x,y) - p_n(x,y)| \le (1+\Lambda_n) \max_{\|(x,y)\|_\infty \le 1} |f(x,y) - p_n^*(x,y)|,$$

$$\Lambda_n(\mathbf{x}_0,\dots,\mathbf{x}_N) = \max_{\mathbf{x}\in[-1,1]^2} \sum_{i=0}^{N} |\ell_i(\mathbf{x})|$$

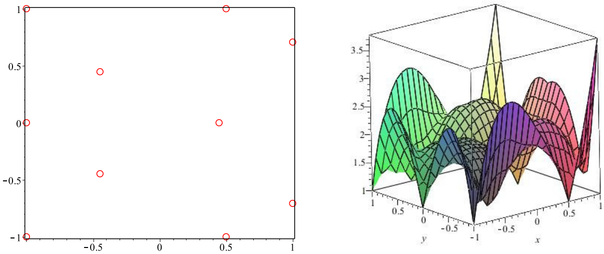
$p_n^* \in \Pi_n^2$ is the best uniform polynomial approximation to f

Order of growth:

- $[-1,1]^2$: Minimal order of growth $\mathcal{O}(\ln^2(n+1))$
  - Padua points [Caliari, De Marchi, Vianello, 2005]
  - Xu points [Bos, De Marchi, Vianello, 2006]

Bivariate polynomial interpolation

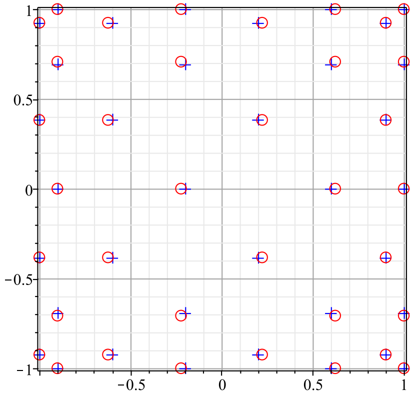
Lebesgue constant on the square [Caliari, De Marchi, Vianello, 2005]



Lebesgue function for 10 Padua points, n=3

Bivariate polynomial interpolation

Mock-Padua points



Graph of 36 Padua points ($\circ$) with their mock-Padua points ($+$) for degree $n = 7$

Bivariate polynomial interpolation

Mock-Padua points

Lebesgue constants of Padua points and mock-Padua points in the unit square

Degree Number of points	1 3	4 15	7 36	10 66	13 105	16 153	19 210
Padua points	2.00	4.41	5.84	6.88	7.71	8.41	9.01
mock-Padua points	2.00	4.56	6.10	6.89	8.07	8.55	9.18

References

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4. Vértesi, P., Optimal Lebesgue constant for Lagrange interpolation, SIAM J. Numer. Anal. 27, 1322–1331, 1990.

THANK YOU FOR YOUR ATTENTION!