Review and complements on dual mixed finite element methods for non-Newtonian fluid flow problems

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Introduction

I– Non–Newtonian fluid flows :

Governed by the classical Stokes problem, the Newtonian fluid flows are a reasonable approximation of the more realistic non–Newtonian fluids (quasi–Newtonian or Viscoelastic).
In the case of **quasi–Newtonian fluids**, the viscosity is a **nonlinear** function of gradient tensor, strain rate tensor, temperature, time, etc.

II- Mixed Formulations :

In the framework of **classical formulations**, one gets **approximations of primal variables** (**velocity, pressure, temperature** $\cdots$).
In addition to the primal variables, the **mixed formulations** enable to obtain **accurate approximations of the dual variables** which are, for the non-Newtonian fluid flow problems,

- **velocity gradient,**
- **the strain rate tensor**,
- **the extra–stress tensor**
- **temperature flux** $\cdots$

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The Non–Newtonian fluid flow problem

• In the case of quasi–Newtonian fluids, the **viscosity** is usually a **Nonlinear function** of gradient tensor, strain rate tensor, temperature, time, etc.

• For a steady and creeping flow of an incompressible quasi-Newtonian fluid, the most used formulation is based on the strain rate tensor, see [Bird et al., Willey (1987)],.

Let $\Omega \subset \mathbb{R}^d$ a bounded domain (of Lipshitz–boundary). The combination of conservation equations leads to the following Nonlinear Stokes problem :

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -\,\boldsymbol{div}\left(2\nu(|\boldsymbol{d}(\boldsymbol{u})|)\,\boldsymbol{d}(\boldsymbol{u})\right) + \nabla p & = & \boldsymbol{f} & \text{in } \Omega, \\ div\,\boldsymbol{u} & = & 0 & \text{in } \Omega, \end{array} \right.$$

where,

• $\boldsymbol{u}$ and p, the **unknowns of the problem**, are respectively, velocity and pressure.
• $\boldsymbol{f}$ is the mass forces.

• $\boldsymbol{d}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\,\boldsymbol{u} + (\nabla\,\boldsymbol{u})^t\right)$ is the strain rate tensor, and

• $|\boldsymbol{d}(\boldsymbol{u})|^2 = \sum_{i,j=1}^{d} \boldsymbol{d}(\boldsymbol{u})_{ij}^2$.

Power law and Carreau model

The **viscosity function** $\nu(\cdot)$, depending on $|\boldsymbol{d}(\boldsymbol{u})|$, is usually given by the two famous following models :

- **Power law :** $\nu(x) = \nu_0\, x^{r-2}, \ \forall x \in \mathbb{R}_+,$
- **Carreau model :** $\nu(x) = \nu_0 \left(1 + x^2\right)^{(r-2)/2}, \ \forall x \in \mathbb{R}_+.$
- $\nu_0 > 0$ is a reference viscosity,
- $1 < r < \infty$ is a characteristic parameter of the fluid.

Finally, system $(\mathcal{P})$ is supplemented by a set of boundary conditions.

Remark

For $r = 2$, both models provide the classical **Stokes problem** :

$$\begin{cases} -\nu_0 \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } \Omega, \\ \mathit{div}\, \boldsymbol{u} &= 0 \quad \text{in } \Omega, \end{cases}$$

corresponding to a **Newtonian** fluid flow.

Ladyzhenskaya model

On the other hand, in connection with the use of the gradient tensor $\nabla \boldsymbol{u}$ which corresponds to the Ladyzhenskaya model :

$$- \boldsymbol{div}\,(2\nu(|\nabla \boldsymbol{u}|)\nabla \boldsymbol{u}) + \nabla p = \boldsymbol{f}, \text{ in } \Omega,$$

$$\nu(|\nabla \boldsymbol{u}|) = (\nu_0 + \nu_1 |\nabla \boldsymbol{u}|)^{r-2}, \ \nu_0 \geq 0, \ \nu_1 > 0, \ r > 1,$$

Remark

The major drawback of formulations using the gradient lies in the fact that we can not deal with **natural boundary conditions**.

Large amount of work is available in the literature. Among these works :

[Du and Gunzburger, SIAM, J. Numer. Ana. 27 (1990)],
[Manouzi and Farhloul, IMA, J. Numer. Ana. 21 (2001)],
[Farhloul and Zine, Comput. Methods Appl. Mech. Engrg. 193 (2002)]
[Gatica et al. IMA J. Numer. Analysis, 23 (2003)],
[Gatica et al. Comput. Methods Appl. Mech. Engrg. 193 (2004)]
[Ervin et al. Comput. Methods Appl. Mech. Engrg. 197 (2008)]

The generalized Stokes problem :

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -\boldsymbol{div}\left(2\nu(|\boldsymbol{d}(\boldsymbol{u})|)\,\boldsymbol{d}(\boldsymbol{u})\right)+\nabla p & = & \boldsymbol{f} & \text{in } \Omega, \\ div\,\boldsymbol{u} & = & 0 & \text{in } \Omega, \\ \text{BC} & & & \text{on } \Gamma \end{array} \right.$$

and its approximation by **standard finite elements** was studied in

[Baranger and Najib, Numer. Math. 58 (1990)]. Extensions and improvements of the error bounds have been obtained in : [Sandri, M^2AN, 27 (1993)], [Barrett and Liu, Numer. Math. 64 (1993), 68 (1994)] ,

• In these works, only the primal variables velocity and pressure are taken into account.

• For various reasons, one may need also information on other physical variables (dual) such as :

- velocity gradients $\nabla\,\boldsymbol{u}$,
- strain rate tensor $\boldsymbol{d}(\boldsymbol{u})$,
- extra-stress tensor $\boldsymbol{\sigma} = 2\nu(|\boldsymbol{d}(\boldsymbol{u})|)\,\boldsymbol{d}(\boldsymbol{u})$.

In this case, it is necessary to build appropriate **mixed formulations**.

Power law : Dual mixed formulation

In the case of Power law, viscosity is given by

$$\nu(|\boldsymbol{d}(\boldsymbol{u})|) = 2\nu_0 \, |\boldsymbol{d}(\boldsymbol{u})|^{r-2} \, \boldsymbol{d}(\boldsymbol{u}).$$

To simplify, we set $\nu_0 = \dfrac{1}{2}$ and define

- a new variable $\boldsymbol{\sigma}$ (the extra-stress tensor) :

$$\boldsymbol{\sigma} = |\boldsymbol{d}(\boldsymbol{u})|^{r-2} \, \boldsymbol{d}(\boldsymbol{u}),$$

- Assuming a homogenous Dirichlet boundary conditions, problem $(\mathcal{P})$ becomes :

$$(\mathcal{P}_m) \left\{ \begin{array}{rcl} \boldsymbol{\sigma} & = & |\boldsymbol{d}(\boldsymbol{u})|^{r-2} \, \boldsymbol{d}(\boldsymbol{u}), \ \text{ in } \Omega, \\ -\,\boldsymbol{div}\,(\boldsymbol{\sigma} - p\,\boldsymbol{I}) & = & \boldsymbol{f}, \ \text{ in } \Omega, \\ div\,\boldsymbol{u} & = & 0, \ \text{ in } \Omega, \\ \boldsymbol{u} & = & 0, \ \text{ on } \Gamma. \end{array} \right.$$

Power law : Dual mixed formulation

In the case of Power law, viscosity is given by

$$\boxed{\nu(|\boldsymbol{d}(\boldsymbol{u})|) = 2\nu_0\, |\boldsymbol{d}(\boldsymbol{u})|^{r-2}\, \boldsymbol{d}(\boldsymbol{u}).}$$

To simplify, we set $\nu_0 = \dfrac{1}{2}$ and define

- a new variable $\boldsymbol{\sigma}$ (the extra-stress tensor) :

$$\boldsymbol{\sigma} = |\boldsymbol{d}(\boldsymbol{u})|^{r-2}\, \boldsymbol{d}(\boldsymbol{u}),$$

- Assuming a homogenous Dirichlet boundary conditions, problem $(\mathcal{P})$ becomes :

$$(\mathcal{P}_m)\begin{cases} \boldsymbol{\sigma} &= |\boldsymbol{d}(\boldsymbol{u})|^{r-2}\, \boldsymbol{d}(\boldsymbol{u}), \quad \text{in } \Omega, \\ -\,\boldsymbol{div}\,(\boldsymbol{\sigma} - p\, \boldsymbol{I}) &= \boldsymbol{f}, \quad \text{in } \Omega, \\ div\, \boldsymbol{u} &= 0, \quad \text{in } \Omega, \\ \boldsymbol{u} &= 0, \quad \text{on } \Gamma. \end{cases}$$

Power law : Dual mixed formulation

Remark (main idea)

From the first equation of $(\mathcal{P}_m)$, on may easily get strain rate tensor $\boldsymbol{d}(\boldsymbol{u})$ as a function of the extra–stress tensor :

$$\boldsymbol{d}(\boldsymbol{u}) = |\boldsymbol{\sigma}|^{r'-2}\,\boldsymbol{\sigma},\ r' \text{ being the conjugate of } r,\quad \frac{1}{r} + \frac{1}{r'} = 1.$$

To simplify the notations we also introduce the operator $\mathcal{A}$ defined by :

$$\mathcal{A}(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}|^{r'-2}\,\boldsymbol{\sigma}.$$

Problem $(\mathcal{P}_m)$ becomes :

$$(\mathcal{P}_m)\begin{cases} \mathcal{A}(\boldsymbol{\sigma}) = \boldsymbol{d}(\boldsymbol{u}), & \text{in } \Omega, \\ -\boldsymbol{div}(\boldsymbol{\sigma} - p\,\boldsymbol{I}) = \boldsymbol{f}, & \text{in } \Omega, \\ div\,\boldsymbol{u} = 0, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \Gamma. \end{cases}$$

Now, the unknowns are $\boldsymbol{u}, p$ and $\boldsymbol{\sigma}$.

[Farhloul and Zine, Numer. Meth. for PDEs 20 (2004)],

[Farhloul and Zine, Int. J. of Numer. Ana. Modeling 5 (2008)],

Power law : Dual mixed formulation

- Assume that $\boldsymbol{f} \in [L^r(\Omega)]^2$, $(1 < r < \infty)$.

- Let $(\boldsymbol{\tau}, q) \in [L^{r'}(\Omega)]^{2\times 2} \times L^{r'}(\Omega)$ and $\boldsymbol{u} \in \left[W^{1,r}(\Omega)\right]^2$ such that $div\, \boldsymbol{u} = 0$.

it easy to see that :

$$\boxed{(\mathcal{A}(\boldsymbol{\sigma}), \boldsymbol{\tau})_r = (\boldsymbol{d}(\boldsymbol{u}), \boldsymbol{\tau})_r = -(\boldsymbol{div}(\boldsymbol{\tau} - q\, \boldsymbol{I}), \boldsymbol{u})_r - (\boldsymbol{\omega}, \boldsymbol{\tau})_r,}$$

$\boldsymbol{\omega} = \frac{1}{2}(\nabla\, \boldsymbol{u} - \nabla^t\, \boldsymbol{u})$ is the vorticity tensor.

$(\cdot, \cdot)_r$ denotes the **duality pairing** between $L^{r'}(\Omega)$ and $L^r(\Omega)$.

The **mixed formulation** is then given by :

$$(MixF1) \begin{cases} \text{Find } \; (\boldsymbol{\sigma}, p) \in \boldsymbol{\Sigma},\; (\boldsymbol{u}, \boldsymbol{\omega}) \in \boldsymbol{M}; \\ (\mathcal{A}(\boldsymbol{\sigma}), \boldsymbol{\tau})_r + (\boldsymbol{div}(\boldsymbol{\tau} - q\, \boldsymbol{I}), \boldsymbol{u})_r + (\boldsymbol{\tau}, \boldsymbol{\omega})_r = 0, \; \forall (\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma}, \\ (\boldsymbol{div}(\boldsymbol{\sigma} - p\, \boldsymbol{I}), \boldsymbol{v})_r + (\boldsymbol{\sigma}, \boldsymbol{\eta})_r + (\boldsymbol{f}, \boldsymbol{v})_r = 0, \; \forall (\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{M}\, . \end{cases}$$

Where

$$\boldsymbol{\Sigma} = \left\{ \underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in [L^{r'}(\Omega)]^{2\times 2} \times L_0^{r'}(\Omega);\; \boldsymbol{div}(\boldsymbol{\tau} - q\, \boldsymbol{I}) \in [L^{r'}(\Omega)]^2 \right\},$$

$$\boldsymbol{M} = \left\{ \underset{\sim}{\boldsymbol{v}} = (\boldsymbol{v}, \boldsymbol{\eta}) \in [L^r(\Omega)]^2 \times [L^r(\Omega)]^{2\times 2};\; \boldsymbol{\eta} + \boldsymbol{\eta}^t = 0 \right\}.$$

equipped with their natural norms : $\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma$ and $\|\underset{\sim}{\boldsymbol{v}}\|_M$.

The previous continuous mixed formulation $(MixF1)$ was studied in :
[Farhloul and Zine, Numer. Methods Partial Differential Equations, 20, 2004].

It is shown there the following results of existence, uniqueness and stability :

Theorem (Existence, uniqueness and stability)

The continuous mixed formulation $(MixF1)$ *admits a unique solution* $(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}}) \in \boldsymbol{\Sigma} \times \boldsymbol{M}$. *Moreover, there is a positive constant* C *depending only on the data such that*

(1) $$\|\underset{\sim}{\boldsymbol{\sigma}}\|_{\Sigma} + \|\underset{\sim}{\boldsymbol{u}}\|_{M} \leq C.$$

Remark

From the second equation of the previous mixed variational formulation,

$$(\boldsymbol{div}(\boldsymbol{\sigma} - p\,\boldsymbol{I}), \boldsymbol{v})_r + (\boldsymbol{\sigma}, \boldsymbol{\eta})_r + (\boldsymbol{f}, \boldsymbol{v})_r = 0, \quad \forall(\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{M},$$

one gets : $(\boldsymbol{\sigma}, \boldsymbol{\eta})_r = 0,\ \forall \boldsymbol{\eta} \in [L^r(\Omega)]^{2\times 2}$ such that $\boldsymbol{\eta} + \boldsymbol{\eta}^t = 0$. This corresponds to the relaxation of symmetry of the extra-stress tensor $\boldsymbol{\sigma}$.

General non–Newtonian fluid flow problem : Dual mixed formulation

Remark

The major drawback of the previous approach is that it cannot be applied to the Carreau Law :

$$\nu(|\boldsymbol{d}(\boldsymbol{u})|) = \left(1 + |\boldsymbol{d}(\boldsymbol{u})|^2\right)^{(r-2)/2}, \quad r \in\,]1, \infty[.$$

• To **avoid** the assumption of expressing the rate of strain tensor as function of the stress tensor and treat both problems associated with **Power law** and **Carreau Model**,

we first formulate problem $(\mathcal{P})$ as :

$$(\mathcal{P}) \begin{cases} \boldsymbol{\sigma} &= \nu(|\boldsymbol{d}(\boldsymbol{u})|\, \boldsymbol{d}(\boldsymbol{u}), \\ -\,\boldsymbol{div}\,(\boldsymbol{\sigma} - p\,\boldsymbol{I}) &= \boldsymbol{f}, \ \text{in } \Omega, \\ div\,\boldsymbol{u} &= 0, \ \text{in } \Omega, \\ \boldsymbol{u} &= 0, \ \text{on } \Gamma. \end{cases}$$

And then, we introduce **two new variables**

$$\boldsymbol{t} = \boldsymbol{d}(\boldsymbol{u}), \quad \text{the strain rate tensor,} \tag{2}$$

$$\boldsymbol{\mathcal{A}}(\boldsymbol{t}) = \nu\,(|\boldsymbol{t}|)\,\boldsymbol{t} = \boldsymbol{\sigma}, \quad \text{the extra stress tensor.} \tag{3}$$

• Now, the unknowns are : $\boldsymbol{u}, p$ and $\boldsymbol{\sigma}, \boldsymbol{t}$.

The dual–mixed formulation of problem $(\mathcal{P})$ reads as follows :

find $\boldsymbol{t} \in \boldsymbol{T}$, $\underset{\sim}{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, p) \in \boldsymbol{\Sigma}$ and $\underset{\sim}{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{\eta}) \in \boldsymbol{M}$, such that :

$$(MixF2)\begin{cases} (\boldsymbol{\mathcal{A}}(\boldsymbol{t}), \boldsymbol{s})_r - (\boldsymbol{\sigma}, \boldsymbol{s})_r = 0 \quad \forall\, \boldsymbol{s} \in \boldsymbol{T}, \\ (\boldsymbol{t}, \boldsymbol{\tau})_r + (\boldsymbol{div}(\boldsymbol{\tau} - q\,\boldsymbol{I}), \boldsymbol{u})_r + (\boldsymbol{\tau}, \boldsymbol{\omega})_r = 0 \quad \forall \underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma}, \\ (\boldsymbol{div}(\boldsymbol{\sigma} - p\,\boldsymbol{I}), \boldsymbol{v})_r + (\boldsymbol{\sigma}, \boldsymbol{\eta})_r + (\boldsymbol{f}, \boldsymbol{v})_r = 0 \quad \forall \underset{\sim}{\boldsymbol{v}} = (\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{M}\,. \end{cases}$$

Where,

$$\boldsymbol{T} = [L^r(\Omega)]^{2\times 2},$$

equipped with its natural norm :

$$\|\boldsymbol{\tau}\|_T = \|\boldsymbol{\tau}\|_{0,r,\Omega} = \left(\int_\Omega |\boldsymbol{\tau}|^r\right)^{\frac{1}{r}}.$$

Remark

Spaces $\boldsymbol{\Sigma}$ and $\boldsymbol{M}$ are the same as for the Power low.

Twofold saddle point problem

To formally rewrite $(MixF2)$ as a **twofold saddle–point problem**, we define the following operators :

$$\boldsymbol{\mathcal{A}} : \boldsymbol{T} \longrightarrow \boldsymbol{T}', \quad \boldsymbol{\mathcal{B}} : \boldsymbol{T} \longrightarrow \boldsymbol{\Sigma}' \quad \text{and} \quad \boldsymbol{\mathcal{C}} : \boldsymbol{\Sigma} \longrightarrow \boldsymbol{M}',$$

$$(4) \qquad \langle \boldsymbol{\mathcal{A}}(\boldsymbol{t}), \boldsymbol{s} \rangle_r = (\boldsymbol{\mathcal{A}}(\boldsymbol{t}), \boldsymbol{s})_r, \ \forall\, \boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{T},$$

$$(5) \qquad \langle \boldsymbol{\mathcal{B}}(\boldsymbol{s}), \underset{\sim}{\boldsymbol{\tau}} \rangle_r = -(\boldsymbol{s}, \boldsymbol{\tau}), \ \forall\, \boldsymbol{s} \in \boldsymbol{T}, \ \forall \underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \boldsymbol{\Sigma},$$

$$(6) \qquad \langle \boldsymbol{\mathcal{C}}(\underset{\sim}{\boldsymbol{\tau}}), \underset{\sim}{\boldsymbol{v}} \rangle_r = -(\boldsymbol{div}(\boldsymbol{\tau} - q\,\boldsymbol{I}), \boldsymbol{v}) - (\boldsymbol{\tau}, \boldsymbol{\eta}), \ \forall \underset{\sim}{\boldsymbol{\tau}} \in \boldsymbol{\Sigma}, \forall \underset{\sim}{\boldsymbol{v}} \in \boldsymbol{M}\,.$$

For a Banach space $\boldsymbol{X}$, $\boldsymbol{X}'$ denotes the dual space with associated norm $\|\cdot\|_{X'}$.

Remark

Recall that the operator $\boldsymbol{\mathcal{A}}$ is defined by

$$\boldsymbol{\mathcal{A}}(\boldsymbol{t}) = \nu(|\boldsymbol{t}|)\ \boldsymbol{t}, \quad \forall\, \boldsymbol{t} \in \boldsymbol{T},$$

ν being given by either **Power** or **Carreau** law.

See
[Gatica et al. IMA J. Numer. Analysis 23 (2003)],
[Gatica et al. Comput. Methods Appl. Mech. Engrg. 193 (2004)]

Using the previous operators $\mathcal{A}(\cdot), \mathcal{B}(\cdot)$ and $\mathcal{C}(\cdot)$, problem $(MixF2)$ is then written in the following **twofold saddle–point** form :

Twofold Saddle–Point Problem

find $\boldsymbol{t} \in \boldsymbol{T}$, $\underset{\sim}{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}, p) \in \boldsymbol{\Sigma}$ and $\underset{\sim}{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{\eta}) \in \boldsymbol{M}$, such that :

$$(MixF2) \begin{cases} \langle \mathcal{A}(\boldsymbol{t}), \boldsymbol{s}\rangle_r + \langle \boldsymbol{s}, \mathcal{B}'(\underset{\sim}{\boldsymbol{\sigma}})\rangle_r &= 0 \quad \forall\, \boldsymbol{s} \in \boldsymbol{T}, \\ \langle \mathcal{B}(\boldsymbol{t}), \underset{\sim}{\boldsymbol{\tau}}\rangle_r + \langle \underset{\sim}{\boldsymbol{\tau}}, \mathcal{C}'(\underset{\sim}{\boldsymbol{u}})\rangle_r &= 0 \quad \forall \underset{\sim}{\boldsymbol{\tau}} \in \boldsymbol{\Sigma}, \\ \langle \mathcal{C}(\underset{\sim}{\boldsymbol{\sigma}}), \underset{\sim}{\boldsymbol{v}}\rangle_r &= \langle \boldsymbol{\mathcal{F}}, \underset{\sim}{\boldsymbol{v}}\rangle_r \quad \forall \underset{\sim}{\boldsymbol{v}} \in \boldsymbol{M}\,. \end{cases}$$

where, $\langle \boldsymbol{\mathcal{F}}, \underset{\sim}{\boldsymbol{v}}\rangle_r = (\boldsymbol{f}, \boldsymbol{v})_r, \forall \underset{\sim}{\boldsymbol{v}} \in \boldsymbol{M},$

$\mathcal{B}'$ and $\mathcal{C}'$ denote the adjoint operators of $\mathcal{B}$ and $\mathcal{C}$ respectively.

Solvability of the continuous problem

Remark

To prove the existence, uniqueness and stability of

$$\left(\boldsymbol{t}, \underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}}\right) = \left(\boldsymbol{t}, (\boldsymbol{\sigma}, p), (\boldsymbol{u}, \boldsymbol{\eta})\right) \in \boldsymbol{T} \times \boldsymbol{\Sigma} \times \boldsymbol{M}$$

solution of $(MixF2)$, we shall recall some technical results given bellow. These results concern the properties of the operators $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$. Mainly :

- $\boldsymbol{\mathcal{A}}$ is bounded, continuous and strictly monotone,
- $\boldsymbol{\mathcal{B}}$ verifies the inf–sup condition on the Kernel of $\boldsymbol{\mathcal{C}}$.
- $\boldsymbol{\mathcal{C}}$ verifies the inf–sup condition.

Here we give several technical lemmas that establish the appropriate conditions on the operators $\boldsymbol{\mathcal{A}}$.

These properties are given for both **Power and Carreau laws**. To distinguish the two models, we set :

- $\delta = 0$ for **Power law**,
- $\delta = 1$ for **Carreau law.**

Remark

The following lemmas, ensure the desired properties : The operator $\boldsymbol{\mathcal{A}}$ is **bounded, continuous and strictly monotone on reflexive Banach spaces.**

Lemma ($1 < r < 2$)

Let $1 < r < 2$. *Then, for all* $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{T} = [L^r(\Omega)]^{2\times 2}$, *we have :*

$$\langle \boldsymbol{\mathcal{A}}(\boldsymbol{s}) - \boldsymbol{\mathcal{A}}(\boldsymbol{t}), \boldsymbol{s} - \boldsymbol{t} \rangle_r \geq C \frac{\| \boldsymbol{s} - \boldsymbol{t} \|_{0,r}^2}{\delta + \| \boldsymbol{s} \|_{0,r}^{2-r} + \| \boldsymbol{t} \|_{0,r}^{2-r}},$$

$$\| \boldsymbol{\mathcal{A}}(\boldsymbol{s}) - \boldsymbol{\mathcal{A}}(\boldsymbol{t}) \|_{0,r'} \leq C \left[\int_\Omega |\boldsymbol{\mathcal{A}}(\boldsymbol{s}) - \boldsymbol{\mathcal{A}}(\boldsymbol{t})| \, |\boldsymbol{s} - \boldsymbol{t}| \, dx \right]^{1/r'}.$$

Lemma ($r \geq 2$)

Let $r \geq 2$. *Then for all* $\boldsymbol{s}, \boldsymbol{t} \in \boldsymbol{T} = [L^r(\Omega)]^{2\times 2}$, *we have :*

$$\langle \boldsymbol{\mathcal{A}}(\boldsymbol{s}) - \boldsymbol{\mathcal{A}}(\boldsymbol{t}), \boldsymbol{s} - \boldsymbol{t} \rangle_r \geq C \left(\| \boldsymbol{s} - \boldsymbol{t} \|_{0,r}^r + \int_\Omega \left(\delta + |\boldsymbol{s}|^{r-2} + |\boldsymbol{t}|^{r-2} \right) |\boldsymbol{s} - \boldsymbol{t}|^2 \, dx \right),$$

$$\begin{aligned} \| \boldsymbol{\mathcal{A}}(\boldsymbol{s}) - \boldsymbol{\mathcal{A}}(\boldsymbol{t}) \|_{0,r'} &\leq C \left[\int_\Omega \left(\delta + |\boldsymbol{s}|^{r-2} + |\boldsymbol{t}|^{r-2} \right) |\boldsymbol{s} - \boldsymbol{t}|^2 \, dx \right]^{1/2} \\ &\times \left(\delta + \| \boldsymbol{s} \|_{0,r}^{(r-2)/2} + \| \boldsymbol{t} \|_{0,\Omega}^{(r-2)/2} \right). \end{aligned}$$

For details on these Lemmas, see [Sandri, M^2AN, 27 (1993)].

And the Browder–Minty theorem.

Lemma 5 : Browder–Minty theorem

- Let $\boldsymbol{T}$ be a **real reflexive Banach space**,
- let $\boldsymbol{\mathcal{A}} : \boldsymbol{T} \longrightarrow \boldsymbol{T}'$ be **bounded, continuous, coercive and monotone.**

Then for any $\boldsymbol{\sigma} \in \boldsymbol{T}'$ there exists a solution $\boldsymbol{t}$ of the equation

$$\boldsymbol{\mathcal{A}}(\boldsymbol{t}) = \boldsymbol{\sigma};$$

i.e. $\boldsymbol{\mathcal{A}}(\boldsymbol{T}) = \boldsymbol{T}'$.

For more details on this Lemma, see [Renardy and Rogers, Theorem 9.45, p. 361 (1993)]

To establish the appropriate inf–sup conditions for the operators $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$, one need some results given in the following Lemmas.

Lemma (4)

Let $(\boldsymbol{X}, \|\cdot\|_X)$ *and* $(\boldsymbol{M}, \|\cdot\|_M)$ *be two reflexive Banach spaces. Let* $\boldsymbol{\mathcal{B}} : \boldsymbol{X} \longrightarrow \boldsymbol{M}'$ *be a linear continuous operator. Let* $\boldsymbol{V} = Ker(\boldsymbol{\mathcal{B}})$ *be the kernel of* $\boldsymbol{\mathcal{B}}$ *; we denote by* $\boldsymbol{V}^o \subset \boldsymbol{X}'$ *the polar set of* $\boldsymbol{V}$*, let* $\dot{\boldsymbol{\mathcal{B}}} : (\boldsymbol{X} \,/\, \boldsymbol{V}) \longrightarrow \boldsymbol{M}'$ *the quotient operator associated to* $\boldsymbol{\mathcal{B}}$*. Then, the three following properties are equivalent :*

(**i**) *there exists* $\beta > 0$*, such that*

$$\inf_{q \in M} \sup_{v \in X} \frac{\langle \boldsymbol{\mathcal{B}}(v), q \rangle_{M', M}}{\|q\|_M \|v\|_X} \geq \beta \,,$$

(**ii**) $\boldsymbol{\mathcal{B}}'$ *is an isomorphism from* $\boldsymbol{M}$ *onto* $\boldsymbol{V}^o$ *and*

$$\|\boldsymbol{\mathcal{B}}' q\|_{X'} \geq \beta \|q\|_M \quad \forall q \in \boldsymbol{M} \,,$$

(**iii**) $\dot{\boldsymbol{\mathcal{B}}}$ *is an isomorphism from* $(\boldsymbol{X} \,/\, \boldsymbol{V})$ *onto* $\boldsymbol{M}'$ *and*

$$\|\dot{\boldsymbol{\mathcal{B}}} \dot{v}\|_{M'} \geq \beta \|\dot{v}\|_{(X/V)} \quad \forall \dot{v} \in (\boldsymbol{X} \,/\, \boldsymbol{V}).$$

The proof of this Lemma is given in [Girault and Raviart, p. 61 (1986)] or [Sandri, M^2AN, 27 (1993)]

Continuous inf–sup conditions for the operators $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$

Here we establish the appropriate inf–sup conditions for the operators $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$.

Define $\boldsymbol{Z}_1$, the null space for $\boldsymbol{\mathcal{C}}$:

$$\boldsymbol{Z}_1 = \left\{ \underset{\sim}{\boldsymbol{\tau}} \in \boldsymbol{\Sigma};\ \langle \boldsymbol{\mathcal{C}}(\underset{\sim}{\boldsymbol{\tau}}), \underset{\sim}{\boldsymbol{v}} \rangle_r = 0,\ \forall \underset{\sim}{\boldsymbol{v}} \in \boldsymbol{M} \right\}.$$

Lemma 6

There exists a constant $\beta_1 > 0$, such that $\displaystyle \inf_{\underset{\sim}{\boldsymbol{\tau}} \in \boldsymbol{Z}_1} \sup_{\boldsymbol{s} \in \boldsymbol{T}} \frac{\langle \boldsymbol{\mathcal{B}}(\boldsymbol{s}), \underset{\sim}{\boldsymbol{\tau}} \rangle_r}{\|\boldsymbol{s}\|_T \ \|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma} \geq \beta_1.$

Lemma 7

There exists a constant $\beta_2 > 0$, such that $\displaystyle \inf_{\underset{\sim}{\boldsymbol{v}} \in \boldsymbol{M}} \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \boldsymbol{\Sigma}} \frac{\langle \boldsymbol{\mathcal{C}}(\underset{\sim}{\boldsymbol{\tau}}), \underset{\sim}{\boldsymbol{v}} \rangle_r}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma \ \|\underset{\sim}{\boldsymbol{v}}\|_M} \geq \beta_2.$

- The proof of Lemma 6 is similar to the ones given in [Manouzi and Farhloul, Lemma 4, IMA, J. Numer. Ana. 21 (2001)]
- To prove Lemma 7, we may use [Farhloul and Zine, proposition 2.2, Numer. Meth. PDEs, 20 (2004)].

Finally, the main result of this section is now presented :

Theorem (Existence, uniqueness and stability of $(MixF2)$)

Problem $(MixF2)$ admits a unique solution $(\boldsymbol{t}, \underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}}) \in \boldsymbol{T} \times \boldsymbol{\Sigma} \times \boldsymbol{M}$ satisfying the following stability condition,

$$\| \, \boldsymbol{t} \, \|_T + \|\underset{\sim}{\boldsymbol{\sigma}}\|_\Sigma + \|\underset{\sim}{\boldsymbol{u}}\|_M \leq C(f).$$

$C(f)$ being a constant depending on f.

Analysis of the discrete dual–mixed formulation

Some finite element notations :

• Let $h > 0$ and $\mathcal{T}_h$ a triangulation of the domain Ω supposed to be polygonal,

• Let $K \in \mathcal{T}_h$, be an element of the triangulation, we denote by $\boldsymbol{b}_K$ the bubble function defined by :

$$\boldsymbol{b}_K(\boldsymbol{x}) = \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})\lambda_3(\boldsymbol{x}),\ \forall \boldsymbol{x} \in K,$$

$\lambda_i, i = 1, 2, 3$ being the barycentric co-ordinates with respect to the element K.

• Let $P_k(K)$ the set of polynomials of degree less than or equal to k on K, and

$$R(K) = \left[P_1(K)\right]^2 \oplus \mathbb{R}\, curl\, \boldsymbol{b}_K.$$

Remark

In practice, the use of $\boldsymbol{d}(\boldsymbol{u})$ introduces a major difficulty related to the symetry of this tensor.
A way to overcome this difficulty is to relax the symetry by a Lagrange multiplier.

To write the **discrete mixed formulation**, we introduce the following finite dimensional sub–spaces :

$$\begin{aligned}
\boldsymbol{T}_h &= \Big\{ \boldsymbol{s}_h \in \boldsymbol{T};\ \boldsymbol{s}_{h|_K} \in R(K),\ \forall K \Big\},\\
\boldsymbol{\Sigma}_h &= \Big\{ \underset{\sim}{\boldsymbol{\tau}_h} = (\boldsymbol{\tau}_h, q_h) \in \boldsymbol{\Sigma};\ \boldsymbol{\tau}_{h|_K} \in [R(K)]^2,\ q_{h|_K} \in P_1(K),\ \forall K \Big\},\\
\boldsymbol{M}_h &= \Big\{ (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in \boldsymbol{M};\ \boldsymbol{v}_{h|_K} \in \big[P_0(K)\big]^2, \boldsymbol{\eta}_h = \theta_h\, \boldsymbol{\chi},\ \ \theta_{h|_K} \in P_1(K), \forall K \Big\}.
\end{aligned}$$

Where,

$$\boldsymbol{\chi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

The discrete mixed formulation is given by :

Discrete Dual Mixed Formulation

$$(MixF2)_h \left\{ \begin{aligned}
\langle \boldsymbol{\mathcal{A}}(\boldsymbol{t}_h), \boldsymbol{s}_h \rangle_r + \langle \boldsymbol{s}_h, \boldsymbol{\mathcal{B}}'(\underset{\sim}{\boldsymbol{\sigma}_h}) \rangle_r &= 0 \ \ \forall\, \boldsymbol{s}_h \in \boldsymbol{T}_h,\\
\langle \boldsymbol{\mathcal{B}}(\boldsymbol{t}_h), \underset{\sim}{\boldsymbol{\tau}_h} \rangle_r + \langle \underset{\sim}{\boldsymbol{\tau}_h}, \boldsymbol{\mathcal{C}}'(\underset{\sim}{\boldsymbol{u}_h}) \rangle_r &= 0 \ \ \forall \underset{\sim}{\boldsymbol{\tau}_h} \in \boldsymbol{\Sigma}_h,\\
\langle \boldsymbol{\mathcal{C}}(\underset{\sim}{\boldsymbol{\sigma}_h}), \underset{\sim}{\boldsymbol{v}_h} \rangle_r &= \langle \boldsymbol{\mathcal{F}}, \underset{\sim}{\boldsymbol{v}_h} \rangle_r \ \ \forall \underset{\sim}{\boldsymbol{v}} \in \boldsymbol{M}.
\end{aligned}\right.$$

Discrete inf–sup conditions for $\boldsymbol{\mathcal{C}}$

As always, for this kind of problem, to show the existence, uniqueness and stability of the solution of problem $(\mathcal{P})_{dm}^{h}$, one should verify that $(\mathcal{P})_{dm}^{h}$ has the same properties as the continuous problem $(\mathcal{P})_{dm}$.

More precisely, one should verify the following properties :

1. $\boldsymbol{\mathcal{C}}$ verifies the discrete inf–sup condition.
2. $\boldsymbol{\mathcal{B}}$ verifies the discrete inf–sup condition on the discrete Kernel of $\boldsymbol{\mathcal{C}}$.

Lemma 8 (Discrete inf–sup for $\mathcal{C}$)

There exists a positive constant β_2^* independent of h, such that

$$\inf_{\underset{\sim}{\boldsymbol{v}_h} \in \boldsymbol{M}_h} \sup_{\underset{\sim}{\boldsymbol{\tau}_h} \in \boldsymbol{\Sigma}_h} \frac{\langle \boldsymbol{\mathcal{C}}(\underset{\sim}{\boldsymbol{\tau}_h}), \underset{\sim}{\boldsymbol{v}_h} \rangle_r}{\|\underset{\sim}{\boldsymbol{\tau}_h}\|_{\Sigma} \, \|\underset{\sim}{\boldsymbol{v}_h}\|_M} \geq \beta_2^*.$$

Discrete inf–sup conditions for $\boldsymbol{\mathcal{B}}$

Define $\boldsymbol{Z}_1^h$ the discrete Kernel of $\boldsymbol{\mathcal{C}}$,

$$\boldsymbol{Z}_1^h = \left\{ \underset{\sim}{\boldsymbol{\tau}_h} \in \boldsymbol{\Sigma}_h ;\ \langle \boldsymbol{\mathcal{C}}(\underset{\sim}{\boldsymbol{\tau}_h}), \underset{\sim}{\boldsymbol{v}_h} \rangle_r = 0,\ \forall \underset{\sim}{\boldsymbol{v}_h} \in \boldsymbol{M}_h \right\}$$

Lemma 9 (Discrete inf–sup for $\boldsymbol{\mathcal{B}}$)

There exists a positive constants β_1^* and C independent of h, such that

$$\inf_{\underset{\sim}{\boldsymbol{\tau}_h} \in \boldsymbol{Z}_1^h} \sup_{\boldsymbol{s}_h \in \boldsymbol{T}_h} \frac{\langle \boldsymbol{\mathcal{B}}(\boldsymbol{s}_h), \underset{\sim}{\boldsymbol{\tau}_h} \rangle_r}{\|\boldsymbol{s}_h\|_T \ \|\underset{\sim}{\boldsymbol{\tau}_h}\|_\Sigma} \geq \beta_1^* .$$

$$\forall \underset{\sim}{\boldsymbol{\tau}_h} \in \boldsymbol{Z}_1^h, \quad \|q_h\|_{0,r'} \leq C \, \|\boldsymbol{\tau}_h\|_{0,r'} \, .$$

The proof of Lemma 9 is given in Proposition 3.1, page 810, and Proposition 3.2, page 811, in

[Farhloul and Zine, Numer. Meth. for PDEs, 20 (2004)]

Finally, du to the previous Lemmas, the inf–sup conditions of $\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{C}}$, we obtain, as for the continuous problem, the existence, uniqueness and stability of the discrete solution. More precisely,

Theorem (Existence, uniqueness and satbility of the discrete solution)

Problem $(MixF2)_h$ admits a unique solution $(\boldsymbol{t}_h, \underset{\sim}{\boldsymbol{\sigma}_h}, \underset{\sim}{\boldsymbol{u}_h}) \in \boldsymbol{T}_h \times \boldsymbol{\Sigma}_h \times \boldsymbol{M}_h$ satisfying the following stability condition,

$$\| \boldsymbol{t}_h \|_T + \|\underset{\sim}{\boldsymbol{\sigma}_h}\|_\Sigma + \|\underset{\sim}{\boldsymbol{u}_h}\|_M \leq C(f).$$

$C(f)$ being a constant depending on f and independent of h.

a priori error estimates

Let :

- $(\boldsymbol{t}, \underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}}) \in \boldsymbol{T} \times \boldsymbol{\Sigma} \times \boldsymbol{M}$ the solution of problem $(MixF2)$,

- $(\boldsymbol{t}_h, \underset{\sim}{\boldsymbol{\sigma}_h}, \underset{\sim}{\boldsymbol{u}_h}) \in \boldsymbol{T}_h \times \boldsymbol{\Sigma}_h \times \boldsymbol{M}_h$ the discrete solution of the problem $(MixF2)_h$.

- For $\boldsymbol{u} \in \left[W^{1,r}(\Omega)\right]$, we first define

$$\boldsymbol{u}_h^* = P_h^0\,\boldsymbol{u}, \text{ the } L^2\text{–projection of } \boldsymbol{u} \text{ onto } \left[\prod_{K \in \boldsymbol{\mathcal{T}}_h} P_0(K)\right]^2.$$

Error estimates : Case $1 < r < 2$

Theorem (Error estimates for $1 < r < 2$)

Let $1 < r < 2$ and $m = 1, 2$. Suppose

- $\boldsymbol{t} \in [W^{m,r}(\Omega)]^{2\times 2}$,
- $\underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\sigma}, p) \in \left[W^{m,r'}(\Omega)\right]^{2\times 2} \times W^{m,r'}(\Omega)$, and
- $\underset{\sim}{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{\omega}) \in [W^{m,r}(\Omega)]^{2} \times [W^{m,r}(\Omega)]^{2\times 2}$.

Then, there exists a positive constant C independent of h such that :

$$\begin{aligned} \|\boldsymbol{t} - \boldsymbol{t}_h\|_{0,r} &\leq C\, h^{m(r/2)}, \\ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r'} + \|p - p_h\|_{0,r'} &\leq C\, h^{m(r-1)}, \\ \|\boldsymbol{u}_h^* - \boldsymbol{u}_h\|_{0,r} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,r} &\leq C\, h^{m(r/2)}. \end{aligned}$$

and

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r} &\leq C\, h^{r/2}, \quad \text{if } m = 1, \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r} &\leq C\, h, \quad \text{if } m = 2. \end{aligned}$$

Error estimates : Case $r \geq 2$

Theorem (Error estimates for $r \geq 2$)

Let $r \geq 2$ and $m = 1, 2$. Suppose

- $\boldsymbol{t} \in [W^{m,r}(\Omega)]^{2\times 2}$,
- $\underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\sigma}, p) \in \left[W^{m,r'}(\Omega)\right]^{2\times 2} \times W^{m,r'}(\Omega)$, and
- $\underset{\sim}{\boldsymbol{u}} = (\boldsymbol{u}, \boldsymbol{\omega}) \in [W^{m,r}(\Omega)]^2 \times [W^{m,r}(\Omega)]^{2\times 2}$.

Then, there exists a positive constant C independent of h such that :

$$\begin{aligned} \|\boldsymbol{t} - \boldsymbol{t}_h\|_{0,r} &\leq C\, h^{m(r'-1)}, \\ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r'} + \|p - p_h\|_{0,r'} &\leq C\, h^{m(r'/2)}, \\ \|\boldsymbol{u}_h^* - \boldsymbol{u}_h\|_{0,r} + \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,r} &\leq C\, h^{m(r'-1)}. \end{aligned}$$

and

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r} &\leq C\, h^{r'-1}, \quad \text{if } m = 1, \\ \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r} &\leq C\, h^{\min\{1, 2(r'-1)\}}, \quad \text{if } m = 2. \end{aligned}$$

Thermally coupled nonlinear Darcy flows

We consider now, an important model in studying the **non-Newtonian flows** with **thermal effects**.
For Ω, a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ, the **thermally coupled nonlinear Darcy flows** :

$$(\mathcal{Q}) \begin{cases} -\boldsymbol{div}(\kappa(T)|\nabla p|^{r-2}\nabla p) = f & \text{in } \Omega, \\ \\ -\Delta T = \kappa(T)|\nabla p|^r & \text{in } \Omega, \\ \\ p = 0 & \text{on } \Gamma, \\ T = 0 & \text{on } \Gamma, \end{cases}$$

where,

- p and T, the unkowns of the problem, denote (respctively) the fluid pressure and the temperature,
- $\kappa(\cdot)$ is the **viscosity depending on temperature**,
- and $1 < r < \infty$.

The **coupling function** $\kappa(\cdot)$ is assumed to be positive bounded measurable, i.e., there exist a positive constants κ^* and $\kappa_* > 0$ such that

$$\kappa^* \geq \kappa_* > 0 \;\text{ and }\; \forall s \in \mathbb{R}, \;\; \kappa_* \leq \kappa(s) \leq \kappa^*.$$

Remark

• Classical formulations of the above problem consist in approximating the **primal variables** :

$$p \quad \text{and} \quad T.$$

• For many reasons, one may need accurate approximation of the **velocity** $\boldsymbol{u}$ and the **heat flux** $\boldsymbol{\xi}$:

$$\begin{cases} \boldsymbol{u} = \kappa(T)|\nabla p|^{r-2}\nabla p \quad \text{and} \\ \boldsymbol{\xi} = \nabla T. \end{cases}$$

[R. P. Gilbert and M. Fang, Math. Comput., 35, pp. 1425-1444 (2002)] .

[J. Zhu, Finite element analysis of thermally coupled nonlinear Darcy flows, Numer. Methods Partial Differential Equations, 26 (2010)].

• Let $r \in]1,\infty[$ and r' its conjugate number. As,

$$\boldsymbol{u} = \kappa(T)|\nabla p|^{r-2}\nabla p,$$

one may obtain ∇p :

$$\boxed{\nabla p = [\kappa(T)]^{1-r'}|\,\boldsymbol{u}\,|^{r'-2}\,\boldsymbol{u} = \mu(T)|\,\boldsymbol{u}\,|^{r'-2}\,\boldsymbol{u}, \qquad \mu(T) = [\kappa(T)]^{1-r'}.}$$

• Assuming that $\mu(\cdot)$ satisfies :

$$\mu_* \le \mu(s) \le \mu^* \ \forall s \in \mathbb{R}, \quad \text{where} \quad \mu_* = [\kappa^*]^{1-r'} \ \text{ and } \ \mu^* = [\kappa_*]^{1-r'}.$$

Introducing new variables $\boldsymbol{u}$ and $\boldsymbol{\xi}$, ystem (Q) is written :

$$(Q_1)\begin{cases} \mu(T)|\,\boldsymbol{u}\,|^{r-2}\,\boldsymbol{u} = \nabla p & \text{in } \Omega, \\ -\,\boldsymbol{div}\,\boldsymbol{u} = f & \text{in } \Omega, \\ & \\ \boldsymbol{\xi} = \nabla T & \text{in } \Omega, \\ -\,div\,\boldsymbol{\xi} = \mu(T)|\,\boldsymbol{u}\,|^{r} & \text{in } \Omega, \\ & \\ p = 0 & \text{on } \Gamma, \\ T = 0 & \text{on } \Gamma. \end{cases}$$

• Let $r \in]1,\infty[$ and r' its conjugate number. As,

$$\boldsymbol{u} = \kappa(T)|\nabla p|^{r-2}\nabla p,$$

one may obtain ∇p :

$$\boxed{\nabla p = [\kappa(T)]^{1-r'}|\,\boldsymbol{u}\,|^{r'-2}\,\boldsymbol{u} = \mu(T)|\,\boldsymbol{u}\,|^{r'-2}\,\boldsymbol{u}, \qquad \mu(T) = [\kappa(T)]^{1-r'}.}$$

• Assuming that $\mu(\cdot)$ satisfies :

$$\mu_* \leq \mu(s) \leq \mu^* \ \forall s \in \mathbb{R}, \quad \text{where} \quad \mu_* = [\kappa^*]^{1-r'} \ \text{ and } \ \mu^* = [\kappa_*]^{1-r'}.$$

Introducing new variables $\boldsymbol{u}$ and $\boldsymbol{\xi}$, ystem (Q) is written :

$$(\mathcal{Q}_1)\begin{cases} \mu(T)|\,\boldsymbol{u}\,|^{r-2}\,\boldsymbol{u} = \nabla p & \text{in } \Omega, \\ -\,\boldsymbol{div}\,\boldsymbol{u} = f & \text{in } \Omega, \\ & \\ \boldsymbol{\xi} = \nabla T & \text{in } \Omega, \\ -\,div\,\boldsymbol{\xi} = \mu(T)|\,\boldsymbol{u}\,|^{r} & \text{in } \Omega, \\ & \\ p = 0 & \text{on } \Gamma, \\ T = 0 & \text{on } \Gamma. \end{cases}$$

Thermally coupled nonlinear Darcy flows : Dual mixed formulation

In order to derive the mixed formulation of problem $(\mathcal{Q}_1)$, we define the following spaces

$$\boldsymbol{V} = \left\{ \boldsymbol{\tau} \in [L^{r'}(\Omega)]^2;\ div\, \boldsymbol{\tau} \in L'(\Omega) \right\}, \qquad \boldsymbol{P} = L^r(\Omega),$$

$$\boldsymbol{Y} = H(div; \Omega), \qquad \boldsymbol{M} = L^2(\Omega).$$

Let $r^* = max\{2, r\}$ and $f \in L^{r^*}(\Omega)$. Our **mixed formulation** of problem $(\mathcal{Q}_1)$ reads as follows :

Find $(\boldsymbol{u}, p, \boldsymbol{\xi}, T) \in \boldsymbol{V} \times \boldsymbol{P} \times \boldsymbol{Y} \times \boldsymbol{M}$ such that

$$(\mathcal{Q}_m) \begin{cases} (\mu(T)|\,\boldsymbol{u}\,|^{r-2}\,\boldsymbol{u}, \boldsymbol{v}) + (div\,\boldsymbol{v}, p) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ (div\,\boldsymbol{u}, q) + (f, q) = 0, \ \forall q \in \boldsymbol{P}, \\ (\boldsymbol{\xi}, \boldsymbol{\eta}) + (div\,\boldsymbol{\eta}, T) = 0, \ \forall \boldsymbol{\eta} \in \boldsymbol{Y}, \\ (div\,\boldsymbol{\xi}, \boldsymbol{\psi}) + (\mu(T)|\,\boldsymbol{u}\,|^r, \boldsymbol{\psi}) = 0, \ \forall \boldsymbol{\psi} \in \boldsymbol{M}. \end{cases}$$

Decoupling procedure : An iterative method

In order to decouple problem $(\mathcal{Q}_m)$, we consider the following iterative method :

- for an arbitrary $T_0 \in \boldsymbol{M}$, and $n = 1, 2, ...$ T_{n-1} being known :

Find $(\boldsymbol{u}_n, p_n, \boldsymbol{\xi}_n, T_n) \in \boldsymbol{V} \times \boldsymbol{P} \times \boldsymbol{Y} \times \boldsymbol{M}$ such that

$$(\mathcal{Q}_{md}) \begin{cases} (\mu(T_{n-1})|\,\boldsymbol{u}_n\,|^{r-2}\,\boldsymbol{u}_n, \boldsymbol{v}) + (div\,\boldsymbol{v}, p_n) = 0 & \forall\,\boldsymbol{v} \in \boldsymbol{V}, \\ (div\,\boldsymbol{u}_n, q) + (f, q) = 0 & \forall q \in \boldsymbol{P}, \\ & \\ (\boldsymbol{\xi}_n, \boldsymbol{\eta}) + (div\,\boldsymbol{\eta}, T_n) = 0 & \forall \boldsymbol{\eta} \in \boldsymbol{Y}, \\ (div\,\boldsymbol{\xi}_n, \boldsymbol{\psi}) + (\mu(T_{n-1})|\,\boldsymbol{u}_n\,|^{r}, \boldsymbol{\psi}) = 0 & \forall \psi \in \boldsymbol{M}\,. \end{cases}$$

Discrete Dual Mixed Formulation

- Suppose that Ω is an open subset of $\mathbb{R}^2$ with **polygonal boundary**.
- Let $\mathcal{T}_h, h > 0$ be a **family of regular triangulations** of Ω into closed triangles.
- Let $P_k(K)$ be the space of polynomials of degree less than or equal to k on $K \in \mathcal{T}_h$.

Let $RT_0(K)$ the lowest degree **Raviart–Thomas element** :

$$RT_0(K) = \Big(P_0(K)\Big)^2 \oplus P_0(K)\,\boldsymbol{x}, \quad \boldsymbol{x} = (x_1, x_2).$$

To approximate the spaces $\boldsymbol{V}, \boldsymbol{P}, \boldsymbol{Y}$ and $\boldsymbol{M}$, we define the following finite–dimensional sub–spaces :

$$\boldsymbol{V}_h = \Big\{ \boldsymbol{v}_h \in \boldsymbol{V};\ \boldsymbol{v}_h|_K \in RT_0(K),\ \forall K \Big\},$$

$$\boldsymbol{P}_h = \Big\{(p \in P;\ p|_K \in P_0(K),\ \forall K \in \mathcal{T}_h \Big\},$$

$$\boldsymbol{Y}_h = \Big\{ \boldsymbol{\eta}_h \in \boldsymbol{Y};\ \boldsymbol{\eta}|_K \in RT_0(K),\ \forall K \in \mathcal{T}_h \Big\},$$

$$\boldsymbol{M}_h = \Big\{ \boldsymbol{\psi}_h \in \boldsymbol{M};\ \boldsymbol{\psi}|_K \in P_0(K),\ \forall K \in \mathcal{T}_h \Big\}.$$

Conclusion and perspectives

This study concerns an analysis of the **non–Newtonian fluid flow** problem and **nonlinear thermal coupling Darcy law.**

Conclusion :

- It avoids the **inversion of the law**, and thus generalize to more realistic models like Carreau model.
- This formulation uses the **symmetric tensor of gradients**, it allows the use of natural boundary conditions.
- The a priori error estimates are **optimal** and correspond to those obtained in the case of the reversal of the power law.
- Mixed Finite Flement formulation of the nonlinear thermal coupling Darcy law giving accurate approximation for both primal and dual variables.
- ...

Perspectives :

- Generalize our approach to viscoelastic fluid flow problems whose solvent portion is constituted by a non-Newtonian fluid.
- Consider non isothermal **quasi–Newtonian** fluid flow problem with viscosity obeying to the **Carreau model** or **Power law**.
- ...

Thank you!

[1] Girault and Raviart, Springer–Verlag (1986)

[2] Bird et al., Willey (1987)

[3] Baranger and Najib, Numer. Math. 58 (1990)

[3] Du and Gunzburger, SIAM, J. Numer. Ana. 27 (1990)

[4] Renardy and Rogers, Springer, NY, (1993)

[5] Sandri, M^2AN, 27 (1993)

[6] Barrett and Liu, Numer. Math. 64 (1993)

[7] Barrett and Liu, Numer. Math. 68 (1994).

[8] Manouzi and Farhloul, IMA, J. Numer. Ana. 21 (2001),

[9] Farhloul and Zine, Comput. Methods Appl. Mech. Engrg. 193 (2002)

[10] Gatica et al. IMA J. Numer. Analysis, 23 (2003)

[11] Farhloul and Zine, Numer. Meth. for PDEs 20 (2004)

[12] Gatica et al. Comput. Methods Appl. Mech. Engrg. 193 (2004)

[13] Farhloul and Zine, Int. J. of Numer. Ana. Modeling 5 (2008)

[14] Ervin et al. Comput. Methods Appl. Mech. Engrg. 197 (2008)

[15] Farhloul and Zine, in writing (2018)