A Fast Local Relaxation Solver for Certain 4th Order PDEs

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Abstract

We consider the following abstract evolution problem

$$\begin{cases} u_t - \operatorname{div}(M(u)\nabla w) = 0, \\ w + E'_-(u) - E'_+(u) + \epsilon^2 \Delta u \in \partial I_S(u), \end{cases} \tag{1}$$

where $M, E_+, E_- : S \to \mathbb{R}$ with $M(u) \geq 0$ and $E_\pm$ convex, $S \subset \mathbb{R}$ is convex (but not necessarily compact). The indicator function $I_S(u)$ is equal to 0 if $u \in S$ and ∞ otherwise, whereas the set-valued function $\partial I_S(u)$ is its subdifferential. Note in particular that (1) implies $u(x) \in S$ almost everywhere. A number of interesting problems can be cast in this framework:

- *Cahn-Hilliard:* $M(u) = 1$, $E_-(u) = \frac{u^2}{2}$, $E_+(u) = \frac{u^4}{4}$ and $S = \mathbb{R}$,
- *Thin films [1]:* $M(u) = \frac{u^3}{3}$, $E_-(u) = 0$, $E_+(u) = 0$ and $S = [0, \infty)$,
- *Deep quench obstacle problem [2]:* $M(u) = 1 - u^2$ (degenerate) or $M(u) = 1$ (non-degenerate), $E_-(u) = \frac{u^2}{2}$, $E_+(u) = 0$ and $S = [-1, 1]$.

Combining $E(u) = E_+(u) - E_-(u)$ (convex-concave splitting) allows us to associate a free energy $\int_\Omega E(u)\,\mathrm{d}x + \frac{\epsilon^2}{2}\|\nabla u\|^2_{L^2}$ with each concrete problem. Our scheme is a reformulation of a semi-implicit FEM scheme introduced in [3] and based on an approximation introduced in [4]. We recast the scheme as a (discrete) convex optimization problem with convex constraints at each time step, and derive a set of relaxation operators that are guaranteed to preserve the total mass and the hard constraint $u \in S$, while reducing the free energy. Furthermore the operators are locally supported and can be applied in parallel, allowing for a highly efficient implementation on modern computer hardware (GPU acceleration). Finally, we discuss the connection of this scheme to a recent study of these type of problems as gradient flows in weighted-Wasserstein metrics [5].

References

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