Precondioned GMRES Method for the Solution of Non-Symmetric Real Toeplitz Systems

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Abstract

It is well known that preconditioned conjugate gradient (PCG) methods are widely used to solve ill-conditioned real symmetric and positive definite Toeplitz linear systems $T_n(f)x = b$. This case has been entirely studied while the case of real non-symmetric and non-definite Toeplitz systems is still open. Toeplitz matrices have the same entries along their diagonals and are generated from the Fourier coefficients of a 2π-periodic generating function or symbol f. Such systems appear in various Mathematical Topics: Differential and Integral equations, Mechanics, Fluid Mechanics and in Applications: signal processing, image processing and restoration, time series, and queueing networks.

The PCG method, fails to solve non-symmetric and/or non-definite systems. In this paper we focus on finding fast and efficient methods, based on Krylov subspaces, to solve large real non-symmetric Toeplitz systems. Especially, we concentrate on the solution of such systems using the Preconditioned Generalized Minimum Residual (PGMRES) method. Real non-symmetric Toeplitz systems are generated by symbols f being complex 2π-periodic functions of the form $f = f_1 + \mathrm{i}f_2$, where f_1 is a 2π-periodic even function while f_2 is a 2π-periodic odd one. If f_1 and f_2 have roots at some points in $[-\pi, \pi]$, then the problem becomes ill-conditioned and some kind of preconditioning is necessary. We propose efficient band Toeplitz preconditioners to solve this kind of problems. The preconditioners are generated by symbols being trigonometric polynomials, which aim at raising the roots and on the other hand at giving some kind of approximation to the functions f_1 and f_2. We achieve a good clustering of the singular values of the preconditioned matrix in a small interval around 1. Also, we discuss on how to generalize the results of the unilevel case to Block Toeplitz systems (2-level), which are generated by 2-variate complex functions.

Finally, we show the efficiency of the proposed technique by presenting various numerical experiments.
