Further insights into the embedding properties of Hadamard matrices

Dimitrios Christou[1], Marilena Mitrouli[2]

[1] *Department of Science and Mathematics, Deree - The American College of Greece, 6 Gravias St., GR-15342 Aghia Paraskevi, Athens, Greece*
email: dchristou@acg.edu

[2] *Department of Mathematics, National and Kapodistrian University of Athens, Panepistemiopolis GR-15773, Athens, Greece*
email: mmitroul@math.uoa.gr

Abstract

Hadamard matrices have many applications in several mathematical areas due to their special form and the numerous properties that characterize them [2, 4]. Based on a recently developed relation between minors of Hadamard matrices [5] and using tools from calculus and elementary number theory, the present work highlights a direct way to investigate the conditions under which an Hadamard matrix of order $n-k$ can or cannot be embedded in an Hadamard matrix of order n. In this study, we analyzed the embedding properties of Hadamard matrices via their minors and revisited the method of proving $H_{n-k} \notin H_n$ when $k < \frac{n}{2}$, which was originally presented in [1]. A systematic approach was followed to this problem, first by looking at the cases $H_{n-4} \notin H_n$ and $H_{n-8} \notin H_n$, and then, considering the general case $H_{n-k} \notin H_n$. The results obtained allowed us to study the problem further when $k \geq \frac{n}{2}$, which may reveal a characteristic embedding pattern for all Hadamard matrices. In particular, for $k = \frac{n}{2}$ it is known that $H_k \in H_{2k}$ and for $k > \frac{n}{2}$ we infer that a Hadamard matrix of order $n-k$ may exist embedded in a Hadamard matrix of order n, that is $H_{n-k} \in H_n$, if the Hadamard matrix of order n has a $k \times k$ submatrix with minor $p\,2^{k-1}$ and the value of p is specifically given by the function:

$$p := \mathcal{P}(n,k) = 2\left(\frac{n}{4}\right)^{\frac{k}{2}} \left(\frac{n-k}{n}\right)^{\frac{n-k}{2}} \quad \text{for} \quad \frac{n}{2} \leq k < n \quad \text{and} \quad \begin{cases} n = 8, 12, 16, \ldots \\ k = 4, 8, 12, \ldots \end{cases}$$

The results obtained also provide answers to the problem of determining the values of the spectrum of the determinant function [3] for specific orders of minors of Hadamard matrices.

References

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