Pseudo-symplectic methods for stochastic Hamiltonian systems

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Abstract

Consider the stochastic autonomous Hamiltonian system in the sense of Stratonovich

$$dP_t^i = -\frac{\partial H_0}{\partial Q_t^i}dt - \sum_{r=1}^{d} \frac{\partial H_r}{\partial Q_t^i} \circ dw_t^r, \quad dQ_t^i = \frac{\partial H_0}{\partial P_t^i}dt + \sum_{r=1}^{d} \frac{\partial H_r}{\partial P_t^i} \circ dw_t^r, \tag{1}$$

where $P_0 = p$, $Q_0 = q$, P, Q, p, q are n-dimensional column vectors, and w_t^r, $r = 1, \ldots, n$ are independent standard Wiener processes. The flow $\phi_t(p,q) = (P_t(p,q), Q_t(p,q))^T$ of (1) preserves the symplectic structure [3]:

$$\left(\frac{\partial \phi_t}{\partial y_0}\right)^T J \left(\frac{\partial \phi_t}{\partial y_0}\right) = J, \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad y_0 = (p,q)^T. \tag{2}$$

Numerical simulations over a long time interval show that symplectic numerical schemes give more accurate approximations for the solutions of stochastic Hamiltonian systems [3]. Several strong and weak symplectic schemes were proposed [3], [1], but unless we consider special stochastic Hamiltonian systems, symplectic schemes are implicit [3]. However, particularly in large Monte Carlo simulations, explicit schemes are desirable in terms of computing time.

For stochastic Hamiltonian systems with additive noise, explicit pseudo- symplectic methods are constructed in [4], and they show good performance for long time computations. In [4] a pseudo-symplectic numerical method $y_1 = \Phi_h(y_0)$, with time step h, of mean-square order (M, N), $N > M$ is defined as a method of mean square order M that satisfies

$$\left(E \left\| \left(\frac{\partial \Phi_h}{\partial y_0}\right)^T J \left(\frac{\partial \Phi_h}{\partial y_0}\right) - J \right\|^2\right)^{1/2} = \mathcal{O}(h^{N+1}). \tag{3}$$

We extend the approach used in the deterministic case [2], and we construct explicit pseudo symplectic schemes in the strong and weak sense for the general stochastic Hamiltonian system (1). We illustrate the properties of these methods thorough several numerical experiments.

References

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