Subspace methods for three-parameter eigenvalue problems

Bor Plestenjak

IMFM and Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia

Abstract

In many applications a PDE has to be solved on a domain that allows the use of the method of separation of variables. In several coordinate systems this leads to two- or three-parameter eigenvalue problems (2EP or 3EP), an example is the Helmholtz equation in ellipsoidal and paraboloidal coordinates. While there exist several subspace methods for 2EP, extensions to 3EP are not straightforward. We propose two subspace methods for 3EP, a subspace iteration with Arnoldi expansion and a Jacobi–Davidson type method, which we generalize from their 2-parameter counterpart and add important new features. Methods are implemented in the Matlab toolbox MultiParEig [1].

In the generic case, separation of variables applied to a separable boundary value problem, followed by a discretization, leads to a multiparameter eigenvalue problem of the form

$$A_{i0}\, x_i = \sum_{j=1}^{k} \lambda_j\, A_{ij}\, x_i, \quad i = 1, \ldots, k, \tag{1}$$

where $k \in \{2, 3\}$ and $A_{ij} \in \mathbb{C}^{n_i \times n_i}$ for $i = 1, \ldots, k$ and $j = 0, \ldots, k$. A k-tuple $(\lambda_1, \ldots, \lambda_k)$ is an eigenvalue if it satisfies (1) for nonzero vectors $x_1, \ldots, x_k$ and the corresponding eigenvector is $x_1 \otimes \cdots \otimes x_k$. By introducing the so-called $k \times k$ operator determinants

$$\Delta_0 = \begin{vmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{k1} & \cdots & A_{kk} \end{vmatrix}_{\otimes} \quad \text{and} \quad \Delta_i = \begin{vmatrix} A_{11} & \cdots & A_{1,i-1} & A_{10} & A_{1,i+1} & \cdots & A_{1k} \\ \vdots & & \vdots & \vdots & & & \vdots \\ A_{k1} & \cdots & A_{k,i-1} & A_{k0} & A_{k,i+1} & \cdots & A_{kk} \end{vmatrix}_{\otimes}$$

for $i = 1, \ldots, k$, where the Kronecker product $\otimes$ is used instead of multiplication, we get matrices $\Delta_0, \ldots, \Delta_k$ of order $n_1 \cdots n_k$. If Δ_0 is nonsingular, then $\Delta_0^{-1}\Delta_1, \ldots, \Delta_0^{-1}\Delta_k$ commute and (1) is equivalent to a system of generalized eigenvalue problems $\Delta_i z = \lambda_i \Delta_0 z$, $i = 1, \ldots, k$, where $z = x_1 \otimes \cdots \otimes x_k$. This relation enables us to compute all eigenvalues of (1) if the Δ-matrices are small. However, usually even for $k = 2$ the Δ-matrices are so large that it is not efficient or even not feasible to compute all the eigenvalues.

In many applications we need only the smallest eigenvalues of $\Delta_k z = \mu \Delta_0 z$. For $k = 2$ we can efficiently apply the Krylov subspace methods since we can exploit the structure of Δ_2 to reduce the complexity of solving a system with Δ_2 from $\mathcal{O}(n_1^3 n_2^3)$ to $\mathcal{O}(n_1^3 + n_2^3)$. For $k = 3$ it remains open how to reduce the complexity of solving a system with Δ_3 below $\mathcal{O}(n_1^3 n_2^3 n_3^3)$. The new methods for 3EP overcome this obstacle and we will show how they can be used to compute accurate eigenfunctions for the ellipsoidal wave equations and Baer wave equations efficiently.

References

[1] B. Plestenjak, MultiParEig, www.mathworks.com/matlabcentral/fileexchange/47844-multipareig